Invariant states on the wreath product

A.V. Dudko, N.I. Nessonov

Abstract

Let $\mathfrak{S}_\infty$ be the infinity permutation group and Γ be a separable topological group. The wreath product $\Gamma \wr \mathfrak{S}_\infty$ is the semidirect product $\Gamma_e^\infty \rtimes \mathfrak{S}_\infty$ for the usual permutation action of $\mathfrak{S}_\infty$ on $\Gamma_e^\infty = \{[\gamma_i]_{i=1}^\infty : \gamma_i \in \Gamma,$ *only finitely many* $\gamma_i \neq e\}$. In this paper we obtain the full description of indecomposable states φ on the group $\Gamma \wr \mathfrak{S}_\infty$, satisfying the condition:

$$\varphi\left(sgs^{-1}\right) = \varphi\left(g\right) \text{ for each } g \in \Gamma \wr \mathfrak{S}_\infty, s \in \mathfrak{S}_\infty.$$

1 Introduction

1.1 The wreath product and $\mathfrak{S}_\infty$-central states. Let $\mathbb{N}$ be the set of the natural numbers. By definition, a bijection $s : \mathbb{N} \to \mathbb{N}$ is called *finite* if the set $\{i \in \mathbb{N} | s(i) \neq i\}$ is finite. Define a group $\mathfrak{S}_\infty$ as the group of all finite bijections $\mathbb{N} \to \mathbb{N}$ and set $\mathfrak{S}_n = \{s \in \mathfrak{S}_\infty | \ s(i) = i \ \text{ for each } i > n\}$. Given a group Γ identify element $(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Gamma^n$ with $(\gamma_1, \gamma_2, \ldots, \gamma_n, e) \in \Gamma^{n+1}$, where e is the identity element of Γ. The group Γ_e^∞ is defined as a inductive limit of sets

$$\Gamma \mapsto \Gamma^2 \mapsto \Gamma^3 \mapsto \cdots \mapsto \Gamma^n \mapsto \cdots . \tag{1}$$

The wreath product $\Gamma \wr \mathfrak{S}_\infty$ is the semidirect product $\Gamma_e^\infty \rtimes \mathfrak{S}_\infty$ for the usual permutation action of $\mathfrak{S}_\infty$ on Γ_e^∞. Using the imbeddings $\gamma \in \Gamma_e^\infty \to (\gamma, \mathrm{id}) \in \Gamma \wr \mathfrak{S}_\infty$, $s \in \mathfrak{S}_\infty \to \left(e^{(\infty)}, s\right) \in \Gamma \wr \mathfrak{S}_\infty$, where $e^{(\infty)} = (e, e, \ldots, e, \ldots)$ and id is the identical bijection, we identify Γ_e^∞ and $\mathfrak{S}_\infty$ with the corresponding subgroups of $\Gamma \wr \mathfrak{S}_\infty$. Therefore, each element g of $\Gamma \wr \mathfrak{S}_\infty$ is of the form $g = s\gamma$, with $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty$ and $s \in \mathfrak{S}_\infty$. Furthermore, it is assumed that $s\left(\gamma_1, \gamma_2, \ldots\right) s^{-1} = \left(\gamma_{s^{-1}(1)}, \gamma_{s^{-1}(2)}, \ldots\right)$.

If Γ is a topological group, then we will equip Γ^n with the natural product-topology. Furthermore, we will always consider Γ_e^∞ as a topological group with the inductive limit topology. The group $\Gamma \wr \mathfrak{S}_\infty$ is isomorphic to $\Gamma_e^\infty \times \mathfrak{S}_\infty$, as a set. Therefore, we will equip the group $\Gamma \wr \mathfrak{S}_\infty$ with the product-topology, considering $\mathfrak{S}_\infty$ as a discrete topological space. From now on we assume that Γ is a separable topological group.

1.2 The basic definitions. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{B}\left(\mathcal{H}\right)$ be the set of all bounded operators in $\mathcal{H}$ and let $\mathcal{I}_\mathcal{H}$ be the identity operator in $\mathcal{H}$. We

denote by $\mathcal{U}(\mathcal{H})$ the unitary subgroup in $\mathcal{B}(\mathcal{H})$. By a unitary representation of the topological group G we will always mean a *continuous* homomorphism of G into $\mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ is equipped with the strong operator topology. For unitary representation π of the group G we denote $\mathcal{M}_\pi$ the W^*-algebra $\pi(G)''$, which is generated by the operators $\pi(g)$ $(g \in G)$.

Definition 1. An unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ of the group G is called a factor-representation if $\mathcal{M}_\pi$ is a factor. A positive definite function φ on group G is called an indecomposable, if the corresponding GNS-representation is a factor-representation.

Further, an element $\Gamma \wr \mathfrak{S}_\infty$ can always be written as the product of an element from $\mathfrak{S}_\infty$ and an element from Γ_e^∞. The commutation rule between these two kinds of elements is

$$s\gamma = s(\gamma_1, \gamma_2, \ldots) = \left(\gamma_{s^{-1}(1)}, \gamma_{s^{-1}(2)}, \ldots\right) s, \tag{2}$$

where $s \in \mathfrak{S}_\infty, \gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty$. Let $\mathbb{N}\diagup s$ be the set of orbits of s on the set $\mathbb{N}$. Note that for $p \in \mathbb{N}\diagup s$ permutation s_p, which is defined by the formula

$$s_p(k) = \begin{cases} s(k) & \text{if } k \in p \\ k & \textit{otherwise} \end{cases}, \tag{3}$$

is a cycle of the order $|p|$, where $|p|$ denotes the cardinality of p. For $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty$ we define the element $\gamma(p) = (\gamma_1(p), \gamma_2(p), \ldots) \in \Gamma_e^\infty$ as follows

$$\gamma_k(p) = \begin{cases} \gamma_k & \text{if } k \in p \\ e & \textit{otherwise.} \end{cases} \tag{4}$$

Thus, using (2), we have

$$s\gamma = \prod_{p \in \mathbb{N}\diagup s} s_p \gamma(p). \tag{5}$$

Element $s_p\gamma(p)$ is called the *generalized cycle* of $s\gamma$.

Denote by $(n\ \ k) \in \mathfrak{S}_\infty$ the transposition of numbers k and n. Following Olshanski (see [3]) we introduce permutations $\omega_n = \omega_n^{(0)} \in \mathfrak{S}_\infty$ by the next formula:

$$\omega_n(i) = \begin{cases} i, & \textit{if } 2n < i, \\ i + n, & \textit{if } i \leqslant n, \\ i - n, & \textit{if } n < i \leqslant 2n. \end{cases} \tag{6}$$

For the element $g = s\gamma$ we call *support* of g the set $\operatorname{supp}(g) = \{i : s(i) \neq i \text{ or } \gamma_i \neq e\}$. Note that $\operatorname{supp}(g)$ is always finite subset of $\mathbb{N}$. If $\operatorname{supp}(g_1) \cap \operatorname{supp}(g_2) = \emptyset$ then elements g_1 and g_2 commute.

Definition 2. Let G be a group and let H be a subgroup of G. A positive definite function φ on G is called H-central if $\varphi(gh) = \varphi(hg)$ for all $h \in H$ and $g \in G$. We say that φ is a *state* on G, if $\varphi(e) = 1$, where e is the identical element of G. A state φ is called *indecomposable*, if the corresponding GNS-representation π_φ is a factor representation.

Let $\mathcal{M}_*$ denotes the space of all σ-weakly continuous functional on w^*-algebra $\mathcal{M}$.

Now we fix a $\mathfrak{S}_\infty$-central state φ on $\Gamma \wr \mathfrak{S}_\infty$, and denote by π_φ the corresponding GNS-representations.

Theorem 3. *Let $\pi_\varphi\left(\Gamma \wr \mathfrak{S}_\infty\right)''$ be a w^*-algebra generated by operators $\pi_\varphi\left(\Gamma \wr \mathfrak{S}_\infty\right)$ and let $\mathcal{C}\left(\pi_\varphi\left(\Gamma \wr \mathfrak{S}_\infty\right)\right)$ be the center of $\pi_\varphi\left(\Gamma \wr \mathfrak{S}_\infty\right)''$. Suppose that the positive functionals φ_1 and φ_2 from $\pi_\varphi\left(\Gamma \wr \mathfrak{S}_\infty\right)''_*$ satisfy the next conditions:*

- i*) $\varphi_k\left(\pi_\varphi(s)a\right) = \varphi_k\left(a\pi_\varphi(s)\right)$ for all $s \in \mathfrak{S}_\infty$ and $a \in \pi_\varphi\left(\Gamma \wr \mathfrak{S}_\infty\right)''$ $(k = 1, 2)$;*
- ii*) $\varphi_1\left(\mathfrak{c}\right) = \varphi_2\left(\mathfrak{c}\right)$ for all $\mathfrak{c} \in \mathcal{C}\left(\pi_\varphi\left(\Gamma \wr \mathfrak{S}_\infty\right)\right)$.*

Then $\varphi_1\left(\mathfrak{a}\right) = \varphi_2\left(\mathfrak{a}\right)$ for all $\mathfrak{a} \in \pi_\varphi\left(\Gamma \wr \mathfrak{S}_\infty\right)$.

Recall that representations π_1 and π_2 of the group G are called quasiequivalent if there exists isomorphism $\theta : \pi_1\left(G\right)'' \mapsto \pi_2\left(G\right)''$ with the property

$$\theta\left(\pi_1\left(g\right)\right) = \pi_2\left(g\right) \text{ for all } g \in G. \tag{7}$$

The following corollary is immediate consequence of the above theorem.

Corollary 4. *If φ_1 and φ_2 are indecomposable $\mathfrak{S}_\infty$-central states on $\Gamma \wr \mathfrak{S}_\infty$ such that the corresponding GNS-representations π_{φ_1} and π_{φ_2} are quasiequivalent, then $\varphi_1 = \varphi_2$.*

1.3 The natural examples.

For any state φ on Γ define two $\mathfrak{S}_\infty$-central states φ_{sp} and φ_{reg} on $\Gamma \wr \mathfrak{S}_\infty$ as follows

$$\varphi_{sp}\left(s\gamma\right) = \prod \varphi\left(\gamma_k\right) \text{ for all } \gamma = \left(\gamma_1, \gamma_2, \ldots\right) \in \Gamma_e^\infty \text{ and } s \in \mathfrak{S}_\infty; \tag{8}$$

$$\varphi_{reg}\left(s\gamma\right) = \begin{cases} \prod \varphi\left(\gamma_k\right) & \text{if } s = e \\ 0 & \text{if } s \neq e. \end{cases} \tag{9}$$

We have the following result:

Proposition 5. *For GNS-representations $\pi_{\varphi_{sp}}$ and $\pi_{\varphi_{reg}}$ the next properties hold:*

- (i) *If $\pi_{\varphi_{sp}}$ acts in Hilbert space $\mathcal{H}_{\varphi_{sp}}$, and $\mathcal{H}_{\varphi_{sp}}^{\mathfrak{S}} = \left\{\eta \in \mathcal{H}_{\varphi_{sp}} : \pi_{sp}(s)\eta = \eta \text{ for all } s \in \mathfrak{S}_\infty\right\}$, then $\dim \mathcal{H}_{\varphi_{sp}}^{\mathfrak{S}} = 1$. In particular, $\pi_{\varphi_{sp}}$ is irreducible.*
- (ii) *$\pi_{\varphi_{reg}}$ is a factor representation.*
- (iii) *w^*-algebra $\pi_{\varphi_{reg}}\left(\Gamma \wr \mathfrak{S}_\infty\right)''$ is a factor of the type* II *or* III.

Proof. Let $\xi_{\varphi_{sp}}$ $\left(\xi_{\varphi_{reg}}\right)$ be the cyclic vector for representation π_{sp} $\left(\pi_{reg}\right)$ with the property

$$\varphi_{sp}\left(g\right) = \left(\pi_{sp}(g)\xi_{\varphi_{sp}}, \xi_{\varphi_{sp}}\right) \qquad \left(\varphi_{reg}\left(g\right) = \left(\pi_{reg}(g)\xi_{\varphi_{reg}}, \xi_{\varphi_{reg}}\right)\right)$$

for all $g \in \Gamma \wr \mathfrak{S}_\infty$.

Set $\Gamma_e^{n\infty} = \left\{\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty \middle| \gamma_k = e \text{ for all } k \leq n\right\}$,
$\mathfrak{S}_{n\infty} = \left\{s \in \mathfrak{S}_\infty \middle| s(k) = k \text{ for all } k \leq n\right\}$. Denote by $\Gamma \wr \mathfrak{S}_{n\infty}$ the subgroup of $\Gamma \wr \mathfrak{S}_\infty$ generated by $\Gamma_e^{n\infty}$ and $\mathfrak{S}_{n\infty}$.

To the proof point (i), first we note that, by definition GNS-construction, $\xi_{\varphi_{sp}}$ lies in $\mathcal{H}_{\varphi_{sp}}^{\mathfrak{S}}$. Further we will use the important mixing-property. Namely, denote by ω_n a bijection which acts as follows

$$\omega_n(i) = \begin{cases} i, & \textit{if } 2n < i, \\ i+n, & \textit{if } i \leqslant n, \\ i-n, & \textit{if } n < i \leqslant 2n. \end{cases} \tag{10}$$

Then for any $\eta \in \mathcal{H}_{\varphi_{sp}}^{\mathfrak{S}}$, using (8), we obtain

$$\lim_{n\to\infty} \left(\pi_{sp}\left(\omega_n\right)\eta, \eta\right) = \left(\xi_{\varphi_{sp}}, \eta\right)\left(\eta, \xi_{\varphi_{sp}}\right). \tag{11}$$

This implies (i).

A property (ii) follows from Proposition 7 (below). Nevertheless, using the explicit realizations of $\pi_{\varphi_{reg}}$, we give another proof. We begin with the GNS-representation T of Γ which acts in Hilbert space $\mathcal{H}_T$ with cyclic vector ξ_φ: $\varphi(\gamma) = (T(\gamma)\xi_\varphi, \xi_\varphi)$ for all $\Gamma \in \gamma$. Further, using embedding $\mathcal{H}_T^{\otimes n} \ni \eta \mapsto \eta \otimes \xi_\varphi \in \mathcal{H}_T^{\otimes n+1}$, define Hilbert space $\mathcal{H}_T^{\otimes\infty}$ and corresponding representation $T^{\otimes\infty}$ of Γ_e^∞:

$$T^{\otimes\infty}(\gamma)\left(\xi_1 \otimes \xi_2 \otimes \ldots\right) = T\left(\gamma_1\right)\xi_1 \otimes T\left(\gamma_2\right)\xi_2 \otimes \ldots, \text{ where } \gamma = \left(\gamma_1, \gamma_2, \ldots\right).$$

The action U of $\mathfrak{S}_\infty$ on $\mathcal{H}_T^{\otimes\infty}$ is given by the formula

$$U(s)\left(\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_k \otimes \ldots\right) = \xi_{s^{-1}(1)} \otimes \xi_{s^{-1}(2)} \otimes \ldots \otimes \xi_{s^{-1}(k)} \otimes \ldots$$

Now we define operator $\Pi(g)$ ($g \in \Gamma \wr \mathfrak{S}_\infty$) in $l^2\left(\mathfrak{S}_\infty, \mathcal{H}_T^{\otimes\infty}\right)$ as follows

$$\left(\Pi(\gamma)\eta\right)(s) = U(s)T^{\otimes\infty}(\gamma)U^*(s)\eta(s) \quad \left(\gamma \in \Gamma_e^\infty, \eta \in l^2\left(\mathfrak{S}_\infty, \mathcal{H}_T^{\otimes\infty}\right)\right);$$
$$\left(\Pi(t)\eta\right)(s) = \eta(st) \quad \left(t \in \mathfrak{S}_\infty\right).$$

Since for any $s \in \mathfrak{S}_\infty$ and $g = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty$ $s\left(\gamma_1, \gamma_2, \ldots\right)s^{-1} = \left(\gamma_{s^{-1}(1)}, \gamma_{s^{-1}(2)}, \ldots\right)$, Π extends by multiplicativity to the representation of $\Gamma \wr \mathfrak{S}_\infty$.

If $\xi_\varphi^{\otimes\infty} = \xi_\varphi \otimes \xi_\varphi \otimes \ldots \in \mathcal{H}_T^{\otimes\infty}$ and $\widehat{\xi}_\varphi(g) = \begin{cases} \xi_\varphi^{\otimes\infty}, & \textit{if } g = e, \\ 0, & \textit{if } g \neq e \end{cases}$ then we have

$$\varphi_{reg}\left(s\gamma\right) = \left(\Pi(s\gamma)\widehat{\xi}_\varphi, \widehat{\xi}_\varphi\right) \quad \left(s \in \mathfrak{S}_\infty, \gamma \in \Gamma_e^\infty\right). \tag{12}$$

Therefore, without loss generality we can assume that $\pi_{reg} = \Pi$.

Let Π' denote the representation of $\mathfrak{S}_\infty$ which acts on $l^2\left(\mathfrak{S}_\infty, \mathcal{H}_T^{\otimes\infty}\right)$ by

$$\left(\Pi'(t)\eta\right)(s) = U(t)\eta(t^{-1}s). \tag{13}$$

Obvious, $\Pi'(\mathfrak{S}_\infty)$ is contained in commutant $\Pi(\Gamma \wr \mathfrak{S}_\infty)'$ of $\Pi(\Gamma \wr \mathfrak{S}_\infty)$.

Let us prove that center $\mathcal{C} = \Pi(\Gamma \wr \mathfrak{S}_\infty)'' \cap \Pi(\Gamma \wr \mathfrak{S}_\infty)'$ of $\Pi(\Gamma \wr \mathfrak{S}_\infty)''$ is trivial.

Our proof starts with the observation that

$$\Pi(g)\Pi'(g)\widehat{\xi}_\varphi = \widehat{\xi}_\varphi \text{ for all } g \in \mathfrak{S}_\infty. \tag{14}$$

Hence for $\mathfrak{c} \in \mathcal{C}$ we have

$$\Pi(g)\Pi'(g)\mathfrak{c}\widehat{\xi}_\varphi = \mathfrak{c}\widehat{\xi}_\varphi \text{ for all } g \in \mathfrak{S}_\infty. \tag{15}$$

In particular, this gives

$$\left\|\mathfrak{c}\widehat{\xi}_\varphi(s)\right\| = \left\|\mathfrak{c}\widehat{\xi}_\varphi\left(gsg^{-1}\right)\right\| \text{ for all } g, s \in \mathfrak{S}_\infty. \tag{16}$$

Since every conjugacy class $C(s) = \left\{gsg^{-1} : g \in \mathfrak{S}_\infty\right\}$ is infinite except $s = e$, we have

$$\mathfrak{c}\widehat{\xi}_\varphi(s) = 0 \text{ for all } s \neq e. \tag{17}$$

It follows from (15) that

$$U(s)\left(\mathfrak{c}\widehat{\xi}_\varphi(e)\right) = \mathfrak{c}\widehat{\xi}_\varphi(e) \text{ for all } s \in \mathfrak{S}_\infty. \tag{18}$$

As in the proof of the point (i), this gives that $\mathfrak{c}\widehat{\xi}_\varphi(e) = \alpha\xi_\varphi^{\otimes\infty}$ $(\alpha \in \mathbb{C})$. Since $\widehat{\xi}_\varphi$ is cyclic, we have $\mathfrak{c} = \alpha I$. Therefore, w^*-algebra $\Pi(\Gamma \wr \mathfrak{S}_\infty)''$ is a factor.

(iii) We begin by recalling the notion of a *central sequence* in a factor $\mathcal{M}$. A bounded sequence $\{a_n\} \subset \mathcal{M}$ is called *central* if

$$s - \lim_{n\to\infty}(a_n m - m a_n) = 0 \text{ and } s - \lim_{n\to\infty}(a_n^* m - m a_n^*) = 0 \text{ for all } m \in \mathcal{M}.$$

A *central sequence* is called *trivial* if there exists sequence $\{c_n\} \subset \mathbb{C}$ such that

$$s - \lim_{n\to\infty}(a_n - c_n I) = 0 \text{ and } s - \lim_{n\to\infty}(a_n^* - \overline{c}_n I) = 0.$$

Let s_k be the transposition interchanging k and $k+1$. We claim that $\{\pi_{reg}(s_n)\}$ is non trivial cental sequence. Indeed, since φ_{reg} is a $\mathfrak{S}_\infty$-central state, we have

$$\lim_{n\to\infty}(m\pi_{reg}(s_n) - \pi_{reg}(s_n)m)\xi_{\varphi_{reg}} = 0 \text{ for all } m \in \Pi(\Gamma \wr \mathfrak{S}_\infty)''.$$

It follows that

$$\lim_{n\to\infty}(m\pi_{reg}(s_n) - \pi_{reg}(s_n)m)x\xi_{\varphi_{reg}} = 0 \text{ for all } m, x \in \Pi(\Gamma \wr \mathfrak{S}_\infty)''.$$

Since $\xi_{\varphi_{reg}}$ is cyclic and $\varphi_{reg}(s_n) = 0$, then $\{\pi_{reg}(s_n)\}$ is non trivial central sequence.

It remains to prove that each cental sequence in factor $\mathcal{M}$ of type I is trivial. Suppose that $\mathcal{M}$ is a factor of type I. Let $\{\mathfrak{e}_{kl} : k, l \in \mathbb{N}\}$ be a matrix unit in $\mathcal{M}$. This means that the next relations hold

$$\mathfrak{e}_{kl}^* = \mathfrak{e}_{lk}, \ \mathfrak{e}_{kl}\mathfrak{e}_{pq} = \delta_{lp}\mathfrak{e}_{kq}, \ \sum_{k\in\mathbb{N}} \mathfrak{e}_{kk} = I. \tag{19}$$

Let $\left\{ a_n = \sum\limits_{k,l} c_{kl}(n)\mathfrak{e}_{kl} : c_{kl}(n) \in \mathbb{C} \right\}$ be a cental sequence in $\mathcal{M}$. Set $\mathfrak{C}_{pq}(n) = a_n\mathfrak{e}_{pq} - \mathfrak{e}_{pq}a_n$. An easy computation shows that

$$\mathfrak{e}_{qq}\left(\mathfrak{C}_{pq}(n)\right)^* \mathfrak{C}_{pq}(n)\mathfrak{e}_{qq} = \left[|c_{pp}(n) - c_{qq}(n)|^2 - |c_{pp}(n)|^2 + \sum_k |c_{kp}(n)|^2 \right] \mathfrak{e}_{qq},$$

$$\mathfrak{e}_{pp}\mathfrak{C}_{pq}(n)\left(\mathfrak{C}_{pq}(n)\right)^* \mathfrak{e}_{pp} = \left[|c_{pp}(n) - c_{qq}(n)|^2 - |c_{qq}(n)|^2 + \sum_k |c_{qk}(n)|^2 \right] \mathfrak{e}_{pp}.$$

Using the fact that $\{a_n\}$ is a central sequence, we deduce from this that

$$\lim_{n\to\infty} \sum_{k:k\neq q} |c_{qk}(n)|^2 = 0, \ \lim_{n\to\infty} \sum_{k:k\neq q} |c_{kq}(n)|^2 = 0,$$
$$\lim_{n\to\infty} |c_{11}(n) - c_{qq}(n)|^2 = 0 \text{ for all } q.$$

This means that $s - \lim\limits_{n\to\infty} (a_n - c_{11}(n)I) = 0$ and $s - \lim\limits_{n\to\infty} \left(a_n^* - \overline{c_{11}(n)}I \right) = 0$. Thus $\{a_n\}$ is trivial. $\square$

The goal of this paper is to give the full description of indecomposable $\mathfrak{S}_\infty$-central states on $\Gamma \wr \mathfrak{S}_\infty$ (see definition 2). The character theory of infinite wreath product in the case of finite Γ is developed by R. Boyer [6]. In this case $\Gamma \wr \mathfrak{S}_\infty$ is inductive limit of finite groups, their finite characters can be obtained as limits of normalized characters of prelimit finite groups, and Boyer's method is a direct generalization of Vershik's-Kerov's asymptotic approach [4]. The characters of $\Gamma \wr \mathfrak{S}_\infty$ for general separable group Γ were found by authors in [9], [10]. Our method has been based on the ideas of Okounkov, which he has developed for the proof of Thoma's theorem [13], [7], [8].

A finite character is a $\Gamma \wr \mathfrak{S}_\infty$-central positive definite function on $\Gamma \wr \mathfrak{S}_\infty$. In this paper we study the more general class of the $\mathfrak{S}_\infty$-central states on $\Gamma \wr \mathfrak{S}_\infty$. Our results provide a complete classification such indecomposable states. The set of all indecomposable $\mathfrak{S}_\infty$-central states have very important property. Namely, if for for two indecomposable $\mathfrak{S}_\infty$-central states φ_1 and φ_2 the corresponding GNS-representations π_{φ_1} and π_{φ_2} are quasiequivalent, then $\varphi_1 = \varphi_2$ (theorem 3, corollary 4).

The papers is organized as follows. Below we give a brief description of the general properties of the $\mathfrak{S}_\infty$-central states. The key results are lemma 6 and proposition 7. Here we also recall the classification of the traces (central

states) on $\Gamma\wr\mathfrak{S}_\infty$ (theorem 9). In section 2 we present the full collection of factor-representations, which define the $\mathfrak{S}_\infty$-central states (proposition 10). Each such state is parametrized by pair (A, ρ), where A is self-adjoint operator, ρ is the unitary representation of Γ (paragraph 2.1). In proposition 11 we prove that the unitary equivalence of pairs (A_1, ρ_1) and (A_2, ρ_2) is equivalent to the equality of the corresponding $\mathfrak{S}_\infty$-central states. In section 3 we discuss about physical KMS-condition (see [15]) for these states (theorem 15). In section 4 we prove the classification theorem 18.

1.4 The multiplicativity. Let φ be an indecomposable $\mathfrak{S}_\infty$-central state on the group $\Gamma\wr\mathfrak{S}_\infty$. Then it defines according to GNS-construction a factor-representation π_φ of the group $\Gamma\wr\mathfrak{S}_\infty$ with cyclic vector ξ_φ such that $\pi_\varphi(g) = (\pi_\varphi(g)\xi_\varphi, \xi_\varphi)$ for each $g \in \Gamma\wr\mathfrak{S}_\infty$. The next lemma shows, that different indecomposable $\mathfrak{S}_\infty$-central states define representations which are not quasiequivalent. Let $w - lim$ stand for the limit in the weak operator topology.

Lemma 6. *Let φ be an indecomposable $\mathfrak{S}_\infty$-central state on the group $\Gamma\wr\mathfrak{S}_\infty$. Than for each $g \in \Gamma\wr\mathfrak{S}_\infty$ there exists $w-\lim\limits_{n\to\infty} \pi_\varphi\left(\omega_n g \omega_n\right)$ and the next equality holds:*

$$w - \lim_{n\to\infty} \pi_\varphi\left(\omega_n g \omega_n\right) = \varphi(g) I. \tag{20}$$

Proof. Let $h_1, h_2 \in \Gamma\wr\mathfrak{S}_\infty$. Fix k such that

$$\operatorname{supp}(h_1), \operatorname{supp}(h_2), \operatorname{supp}(g) \subset \{1, 2, \ldots, k\}. \tag{21}$$

For each $n \in \mathbb{N}$ there exists elements $g_{(n,k)}, h_{(n,k)} \in \mathfrak{S}_\infty$ such that

$$\operatorname{supp}(g_{(n,k)}), \operatorname{supp}(h_{(n,k)}) \subset \{k+1, k+2, \ldots\} \tag{22}$$

and $\omega_{n+k} = g_{(n,k)}\omega_k h_{(n,k)}$ (see (6)). Permutations $g_{(n,k)}, h_{(n,k)}$ can be defined as follows:

$$g_{(n,k)}(i) = \begin{cases} i, & \textit{if } \ i \leqslant k \ \textit{ or } 2k+2n < i, \\ i+n, & \textit{if } \ k < i \leqslant 2k+n, \\ i-k-n, & \textit{if } \ 2k+n < i \leqslant 2k+2n. \end{cases}$$

$$h_{(n,k)}(i) = \begin{cases} i, & \textit{if } \ i \leqslant k \ \textit{ or } 2k+n < i, \\ i+k, & \textit{if } \ k < i \leqslant k+n, \\ i-n, & \textit{if } \ k+n < i \leqslant 2k+n. \end{cases}$$

By (21) and (22), the elements $g_{(n,k)}$ and $h_{(n,k)}$ commutes with the elements h_1, h_2 and g. Therefore

$$\begin{aligned} h_2^{-1}\omega_{n+k} g \omega_{n+k} h_1 &= h_2^{-1}\left(g_{(n,k)}\omega_k h_{(n,k)}\right)^{-1} g g_{(n,k)}\omega_k h_{(n,k)} h_1 \\ &= h_{(n,k)}^{-1} h_2^{-1}\omega_k g \omega_k h_1 h_{(n,k)}. \end{aligned} \tag{23}$$

As φ is $\mathfrak{S}_\infty$-central, one has:

$$\begin{aligned} &(\pi_\varphi\left(\omega_{n+k} g \omega_{n+k}\right) \pi_\varphi(h_1)\xi_\varphi, \pi_\varphi(h_2)\xi_\varphi) = \varphi\left(h_2^{-1}\omega_{n+k} g \omega_{n+k} h_1\right) = \\ &\varphi\left(h_2^{-1}\omega_k g \omega_k h_1\right) = (\pi_\varphi\left(\omega_k g \omega_k\right) \pi_\varphi(h_1)\xi_\varphi, \pi_\varphi(h_2)\xi_\varphi). \end{aligned} \tag{24}$$

As ξ_φ is cyclic, by (24), there exists the limit

$$w - \lim_{n\to\infty} \pi_\varphi(\omega_n g \omega_n).$$

For each $h \in \Gamma \wr \mathfrak{S}_\infty$ for large enough n one has $\operatorname{supp}(\omega_n g \omega_n) \cap \operatorname{supp}(h) = \emptyset$. Therefore $\pi_\varphi(\omega_n g \omega_n)\pi_\varphi(h) = \pi_\varphi(h)\pi_\varphi(\omega_n g \omega_n)$. This involves that the weak limit $w - \lim\limits_{n\to\infty} \pi_\varphi(\omega_n g \omega_n)$ lies in the center of the algebra M_{π_φ}, generated by operators of the representation π_φ. Thus $\lim\limits_{n\to\infty} \pi_\varphi(\omega_n g \omega_n)$ is scalar. By $\mathfrak{S}_\infty$-centrality of φ,

$$\left(w - \lim_{n\to\infty} \pi_\varphi(\omega_n g \omega_n)\xi_\varphi, \xi_\varphi\right) = \lim_{n\to\infty} \varphi(\omega_n g \omega_n) = \varphi(g),$$

which finishes the proof. $\square$

The following claim gives a useful characterization of the class of the indecomposable $\mathfrak{S}_\infty$-central states:

Proposition 7. *The following conditions for $\mathfrak{S}_\infty$-central state φ on the group $\Gamma \wr \mathfrak{S}_\infty$ are equivalent:*

- *(a) φ is indecomposable;*
- *(b) $\varphi(gg') = \varphi(g)\varphi(g')$ for each $g, g' \in \Gamma \wr \mathfrak{S}_\infty$ with $\operatorname{supp}(g) \cap \operatorname{supp}(g') = \emptyset$;*
- *(c) $\varphi(g) = \prod\limits_{p\in\mathbb{N}/s} \varphi\left(s_p\gamma(p)\right)$ for each $g = s\gamma = \prod\limits_{p\in\mathbb{N}/s} s_p\gamma(p)$ (see 5).*

Proof. The equivalence of (b) and (c) is obvious. We prove the equivalence of (a) and (b). Using GNS-construction, we build the representation π_φ of the group $\Gamma \wr \mathfrak{S}_\infty$ which acts in the Hilbert space $\mathcal{H}_\varphi$ with cyclic vector ξ_φ such that

$$\varphi(g) = \left(\pi_\varphi\left(g\right)\xi_\varphi, \xi_\varphi\right) \text{ for each } g \in \Gamma \wr \mathfrak{S}_\infty.$$

Suppose that the property (a) holds. Consider two elements $g = s\gamma$ and $g' = s'\gamma'$ from $\Gamma \wr \mathfrak{S}_\infty$ satisfying $\operatorname{supp}(g) \cap \operatorname{supp}(g') = \emptyset$. Then there exists a sequence $\{s_n\}_{n\in\mathbb{N}} \subset \mathfrak{S}_\infty$ such that for each n

$$\operatorname{supp}(s_n) \cap \operatorname{supp}(g) = \emptyset \text{ and } \operatorname{supp}(s_n g' s_n^{-1}) \subset \{n+1, n+2, \ldots\}. \tag{25}$$

For example we can put $s_n = \prod\limits_{i\in\operatorname{supp}(g')} (i, i+k+n)$, where k is fixed number such that $\operatorname{supp}(g) \cup \operatorname{supp}(g') \subset \{1, 2, \ldots, k\}$. Using the ideas of the proof of the

lemma 6 we obtain, that the limit $\lim\limits_{n\to\infty} \pi_\varphi(s_n g' s_n)$ exists in the weak operator topology and the next equality holds:

$$w - \lim_{n\to\infty} \pi_\varphi(s_n g' s_n) = \varphi(g') I. \tag{26}$$

Using (25), (26) and $\mathfrak{S}_\infty$-centrality of φ, we obtain

$$\varphi\left(gg'\right) = \lim_{n\to\infty} \varphi\left(g s_n g' s_n^{-1}\right) =$$
$$\lim_{n\to\infty} \left(\pi_\varphi(g)\pi_\varphi\left(s_n g' s_n^{-1}\right)\xi_\varphi, \xi_\varphi\right) = \varphi(g)\varphi\left(g'\right).$$

Thus *(b)* follows from *(a)*.

Further suppose that the condition *(b)* holds. If $\pi_\varphi\left(\Gamma \wr \mathfrak{S}_\infty\right)' \bigcap \pi_\varphi\left(\Gamma \wr \mathfrak{S}_\infty\right)'' = \mathcal{Z}$ is larger than the scalars, then it contains a pair of orthogonal projections E and F satisfying the condition:

$$EF = 0. \tag{27}$$

Fix arbitrary $\varepsilon > 0$. By the von Neumann Double Commutant Theorem there exist $g_k, h_k \in \Gamma \wr \mathfrak{S}_\infty$ and complex numbers c_k, d_k $(k = 1, 2, \ldots, N < \infty)$ such that

$$\begin{aligned} \left\|\sum_{k=1}^N c_k \pi_\varphi\left(g_k\right)\xi_\varphi - E\xi_\varphi\right\| < \varepsilon, \\ \left\|\sum_{k=1}^N d_k \pi_\varphi\left(h_k\right)\xi_\varphi - F\xi_\varphi\right\| < \varepsilon. \end{aligned} \tag{28}$$

Fix n such that $\operatorname{supp}(g_k) \subset \{1, 2, \ldots, n\}$ and $\operatorname{supp}(h_k) \subset \{1, 2, \ldots, n\}$ for each k. As φ is $\mathfrak{S}_\infty$-central, using (28), we obtain

$$\left\|\sum_{k=1}^N c_k \pi_\varphi\left(\omega_n g_k \omega_n\right)\xi_\varphi - E\xi_\varphi\right\| < \varepsilon, \ \textit{(see (6))}. \tag{29}$$

Now, using (27), (28) and (29), we have

$$\left|\left(\sum_{k=1}^N c_k \pi_\varphi\left(\omega_n g_k \omega_n\right)\sum_{k=1}^N d_k \pi_\varphi\left(h_k\right)\xi_\varphi, \xi_\varphi\right)\right| < 2\varepsilon + \varepsilon^2. \tag{30}$$

Note, that $\operatorname{supp}(\omega_n g_k \omega_n) \subset \{n+1, n+2, \ldots\}$ for each k. Therefore, by the property (b), (28) and (29), one has:

$$\begin{gathered} \left|\left(\sum_{k=1}^N c_k \pi_\varphi\left(\omega_n g_k \omega_n\right)\sum_{k=1}^N d_k \pi_\varphi\left(h_k\right)\xi_\varphi, \xi_\varphi\right)\right| = \\ \left|\left(\sum_{k=1}^N c_k \pi_\varphi\left(\omega_n g_k \omega_n\right)\xi_\varphi, \xi_\varphi\right)\left(\sum_{k=1}^N d_k \pi_\varphi\left(h_k\right)\xi_\varphi, \xi_\varphi\right)\right| > \\ \left(E\xi_\varphi, \xi_\varphi\right)\left(F\xi_\varphi, \xi_\varphi\right) - \varepsilon\left(\left(E\xi_\varphi, \xi_\varphi\right) + \left(F\xi_\varphi, \xi_\varphi\right)\right) - \varepsilon^2. \end{gathered} \tag{31}$$

Note that, as ξ_φ is cyclic, $E\xi_\varphi \neq 0$ and $F\xi_\varphi \neq 0$. Therefore, taking in view (30) and (31), we arrive at a contradiction. $\square$

Denote the element $\sigma_n \in \mathfrak{S}_\infty$ by the formula:

$$\sigma_n(i) = \begin{cases} i+1 & \textit{if } i < n, \\ 1 & \textit{if } i = n, \\ i & \textit{if } i > n. \end{cases} \tag{32}$$

Corollary 8. *Each indecomposable $\mathfrak{S}_\infty$-central state φ on the group $\Gamma \wr \mathfrak{S}_\infty$ is defined by its values on the elements of the form $\sigma_n\gamma$, where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n, e, e, \ldots)$ and $n \in \mathbb{N}$.*

Proof. By the proposition 7, φ is defined by its values on the elements of the view $s_p\gamma(p)$ (see (5)). Fix an element $s_p\gamma(p)$. Let $n = |p|$. Then there exists a permutation $h \in \mathfrak{S}_\infty$ such that $hs_ph^{-1} = \sigma_n$. Therefore $\varphi(s_p\gamma(p)) = \varphi(hs_p\gamma(p)h^{-1}) = \varphi(\sigma_n h\gamma(p)h^{-1})$, which proves the corollary. $\square$

1.5 The characters of the group $\mathfrak{S}_\infty$ and $\Gamma \wr \mathfrak{S}_\infty$. In the paper [13], E.Thoma obtained the following remarkable description of all *indecomposable* character ($\mathfrak{S}_\infty$-central states) of the group $\mathfrak{S}_\infty$. Characters of the group $\mathfrak{S}_\infty$ are labeled by a pair of non-increasing positive sequences of numbers $\{\alpha_k\}$, $\{\beta_k\}$ ($k \in \mathbb{N}$), such that

$$\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \beta_k \leq 1. \tag{33}$$

The value of the corresponding character on a cycle of length l is

$$\sum_{k=1}^{\infty} \alpha_k^l + (-1)^{l-1} \sum_{k=1}^{\infty} \beta_k^l.$$

Its value on a product of several disjoint cycles equals to the product of values on each of cycles.

In [9] authors described all indecomposable characters on the group $\Gamma \wr \mathfrak{S}_\infty$. Before to formulate the main result of [9] we introduce some more notations. We call an element $g = s\gamma$ a generated cycle if either s is a cycle and $\operatorname{supp}(\gamma) \subset \operatorname{supp}(s)$ or $s = e$ and $\operatorname{supp}(\gamma) = \{n\}$ for some n. For an element $g = s\gamma$ and an orbit $p \in \mathbb{N}/s$ choose the minimal number $k \in p$ and denote

$$\tilde{\gamma}(p) = \gamma_k \gamma_{s^{(-1)}(k)} \cdots \gamma_{s^{(-l)}(k)} \cdots \gamma_{s^{(-|p|+1)}(k)}. \tag{34}$$

For a factor-representation τ of the finite type let χ_τ be its normalized character. That is $\chi_\tau(g) = tr_{\mathcal{M}_\tau}(\tau(g))$, where $tr_{\mathcal{M}}$ stands for the unique normal, normalized ($tr_{\mathcal{M}}(I) = 1$) trace on the factor $\mathcal{M}$ of the finite type. Note that $\chi_\tau(e) = 1$. Let tr be the ordinary matrix normalized trace.

Theorem 9 ([9], [10]). *Let φ be a function on the group $\Gamma \wr \mathfrak{S}_\infty$. Then the following conditions are equivalent.*

a) φ is an indecomposable character.

b) There exist a representation τ of the finite type of the group Γ, two non-increasing positive sequences of numbers $\{\alpha_k\}$, $\{\beta_k\}$ $(k \in \mathbb{N})$ and two sequences $\{\rho_k\}, \{\varrho_k\}$ of finite-dimensional irreducible representations of Γ with properties

- (i) $\delta = 1 - \sum\limits_k \alpha_k dim\, \rho_k - \sum\limits_k \beta_k dim \varrho_k \geqslant 0$;
- (ii) *if s is cycle, $g = s\gamma$ $(\gamma \in \Gamma_e^\infty)$, $p = \mathrm{supp} s = \mathrm{supp}\,(s\gamma)$, then*

$$\varphi(g) = \begin{cases} \sum\limits_k \alpha_k\, tr(\rho_k(\gamma_n)) + \sum\limits_k \beta_k,\, tr(\varrho_k(\gamma_n)) + \delta\chi_\tau(\gamma_n), \;\; if \;\; p = \{n\}, \\ \sum\limits_k \alpha_k^{|p|} tr(\rho_k(\tilde{\gamma}(p))) + (-1)^{|p\,|-1} \sum\limits_k \beta_k^{|p\,|} tr(\varrho_k(\tilde{\gamma}(p))), \;\; if \;\; |p\,| > 1; \end{cases}$$

- (iii) *if $g = s\gamma = \prod\limits_{p \in \mathbb{N}/s} s_p\gamma(p)$ (see 5), then $\varphi(g) = \prod\limits_{p \in \mathbb{N}/s} \varphi\,(s_p\gamma(p))$.*

2 Examples of representations.

2.1 Parameters of states. Let A be a self-adjoint operator of the *trace class* (see [12]) from $\mathcal{B}(\mathcal{H})$ with the property:

$\mathrm{Tr}(|A|) \leq 1$, where Tr is ordinary trace[1] on $\mathcal{B}(\mathcal{H})$.

Further we fix vector $\hat{\xi} \in \mathrm{Ker}\, A$ and the unitary representation ρ of Γ in $\mathcal{H}$, which satisfies the conditions:

- (1) if $\mathrm{Tr}(|A|) = 1$, then subspace $(\mathrm{Ker}\, A)^\perp = \mathcal{H} \ominus \mathrm{Ker}\, A$ is cyclic for w^*-algebra $\mathfrak{A}$ generated by A and $\rho(\Gamma)$;
- (2) if $\mathrm{Tr}(|A|) < 1$, subspace $\widetilde{\mathcal{H}}$ is generated by $\left\{\mathfrak{A}v, v \in (\mathrm{Ker}A)^\perp\right\}$ and $\mathcal{H}_{reg} = \mathcal{H} \ominus \widetilde{\mathcal{H}}$, then $\dim \mathcal{H}_{reg} = \infty$;
- (3) if $P_{]0,1]}$ and $P_{[-1,0[}$ are the spectral projections of A, then subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ generated by vectors $\left\{\mathfrak{A}v,\ v \in P_{]0,1]}\mathcal{H}\right\}$ and $\left\{\mathfrak{A}v,\ v \in P_{[-1,0[}\mathcal{H}\right\}$, respectively, are orthogonal;
- (4) there exist I_∞-factor $N'_{reg} \subset \left(\rho\,(\Gamma)\Big|_{\mathcal{H}_{reg}}\right)'$ with matrix unit $\{\mathfrak{e}'_{kl},\ k, l \in \mathbb{N}\}$ such that $\hat{\xi} \in \mathfrak{e}'_{11}\mathcal{H}_{reg}$, $\left\|\hat{\xi}\right\| = 1$ and $\mathfrak{e}'_{11}\mathcal{H}_{reg}$ is generated by $\left\{\rho\,(\Gamma)\,\hat{\xi}\right\}$. In particular, if $\mathrm{Tr}(|A|) = 1$ then $\hat{\xi} = 0$. When $\mathrm{Tr}(|A|) < 1$ we assume for convenience that $\left\|\hat{\xi}\right\| = 1$.

[1] If $\mathfrak{p}$ is the minimal projection from $\mathcal{B}(\mathcal{H})$, then $\mathrm{Tr}(\mathfrak{p}) = 1$.

2.2 Hilbert space $\mathcal{H}_A^\rho$. Define a state ψ_k on $\mathcal{B}(\mathcal{H})$ as follows

$$\psi_k(v) = \mathrm{Tr}(v|A|) + (1 - \mathrm{Tr}(|A|)) \left(v\mathfrak{e}'_{k1}\hat{\xi}, \mathfrak{e}'_{k1}\hat{\xi} \right), \quad v \in \mathcal{B}(\mathcal{H}). \tag{35}$$

Let ${}_1\psi_k$ denote the product-state on $\mathcal{B}(H)^{\otimes k}$:

$${}_1\psi_k(v_1 \otimes v_2 \otimes \ldots \otimes v_k) = \prod_{j=1}^{k} \psi_j(v_j). \tag{36}$$

Now define inner product on $\mathcal{B}(H)^{\otimes k}$ by

$$(v, u)_k = {}_1\psi_k(u^* v). \tag{37}$$

Let $\mathcal{H}_k$ denote the Hilbert space obtained by completing $\mathcal{B}(H)^{\otimes k}$ in above inner product norm. Now we consider the natural isometrical embedding

$$v \ni \mathcal{H}_k \mapsto v \otimes \mathrm{I} \in \mathcal{H}_{k+1}. \tag{38}$$

and define Hilbert space $\mathcal{H}_A^\rho$ as completing $\bigcup\limits_{k=1}^{\infty} \mathcal{H}_k$.

2.3 The action $\Gamma \wr \mathfrak{S}_\infty$ on $\mathcal{H}_A^\rho$. First, using the embedding $a \ni \mathcal{B}(\mathcal{H})^{\otimes k} \mapsto a \otimes \mathrm{I} \in \mathcal{B}(\mathcal{H})^{\otimes k+1}$, we identify $\mathcal{B}(\mathcal{H})^{\otimes k}$ with subalgebra $\mathcal{B}(\mathcal{H})^{\otimes k} \otimes \mathbb{C} \subset \mathcal{B}(\mathcal{H})^{\otimes k+1}$. Therefore, algebra $\mathcal{B}(\mathcal{H})^{\otimes \infty} = \bigcup\limits_{n=1}^{\infty} \mathcal{B}(\mathcal{H})^{\otimes n}$ is well defined.

Further we give the explicit embedding $\mathfrak{S}_\infty$ into unitary group of $\mathcal{B}(\mathcal{H})^{\otimes \infty}$. First fix the matrix unit $\{e_{pq} : p, q = 1, 2, \ldots, n = \dim \mathcal{H}\} \subset \mathcal{B}(\mathcal{H})$ with the properties:

- (i) projection e_{kk} is minimal and $e_{kk}A = c_{kk}e_{kk}$ ($c_{kk} \in \mathbb{C}$) for all $k = 1, 2, \ldots, n$;
- (ii) $e_{kk}\mathcal{H}_+ \subset \mathcal{H}_+$ and $e_{kk}\mathcal{H}_- \subset \mathcal{H}_-$ for all $k = 1, 2, \ldots, n$.

Put $X = \{1, 2, \ldots, n\}^{\times\infty}$. For $x = (x_1, x_2, \ldots, x_l, \ldots) \in X$ we set $\mathfrak{l}_A(x) = |\{i : e_{x_i\, x_i}\mathcal{H} \subset \mathcal{H}_-\}|$. Define subsequence $x_A = (x_{i_1}, x_{i_2}, \ldots x_{i_l}, \ldots) \in \{1, 2, \ldots, n\}^{\mathfrak{l}_A(x)}$ by induction

$$i_1 = \min\{i : e_{x_i\, x_i}\mathcal{H} \subset \mathcal{H}_-\} \text{ and } i_k = \min\{i > i_{k-1} : e_{x_i\, x_i}\mathcal{H} \subset \mathcal{H}_-\}. \tag{39}$$

For $s \in \mathfrak{S}_\infty$ denote by $c(x, s)$ the unique permutation from $\mathfrak{S}_{\mathfrak{l}_A(x)} \subset \mathfrak{S}_\infty$ such that

$$s^{-1}\left(i_{c(x,s)(1)}\right) < s^{-1}\left(i_{c(x,s)(2)}\right) < \ldots < s^{-1}\left(i_{c(x,s)(l)}\right) < \ldots. \tag{40}$$

Let $\mathfrak{S}_\infty$ acts on X as follows

$$X \times \mathfrak{S}_\infty \ni (x, s) \mapsto sx = \left(x_{s(1)}, x_{s(2)}, \ldots, x_{s(l)}, \ldots\right) \in X. \tag{41}$$

By definition, $(sx)_A = \left(x_{i_{c(x,s)(1)}}, x_{i_{c(x,s)(2)}}, \ldots, x_{i_{c(x,s)(l)}}, \ldots\right)$. Therefore,

$$c(x,ts) = c(sx,t)c(x,s) \quad \text{for all } \ t,s \in \mathfrak{S}_\infty; x \in X. \tag{42}$$

Given any $s \in \mathfrak{S}_\infty$ put

$$U_N(s) = \sum_{x_1,x_2,\ldots,x_N=1}^{n} \text{sign } (c(x,s)) \ e_{x_{s(1)}\,x_1} \otimes e_{x_{s(2)}\,x_2} \otimes \ldots \otimes e_{x_{s(N)}\,x_N},$$

where $N < \infty$ satisfies the condition: $s(i) = i$ for all $i \geq N$, $x = (x_1, x_2, \ldots, x_N, \ldots)$. We see at once that for $L > N$

$$U_N(s) \otimes \underbrace{\mathrm{I} \otimes \mathrm{I} \otimes \ldots \otimes \mathrm{I}}_{L-N} = U_L(s).$$

Thus operator $U(s) = U_N(s) \otimes \mathrm{I} \otimes \mathrm{I} \otimes \ldots \in \mathcal{B}\left(\mathcal{H}\right)^{\otimes\infty} = \bigcup\limits_{n=1}^{\infty} \mathcal{B}\left(\mathcal{H}\right)^{\otimes n}$ is well defined. It follows from 42 that

$$U(t)U(s) = U(ts) \text{ for all } t,s \in \mathfrak{S}_\infty. \tag{43}$$

It is clear that

$$\begin{aligned}&\text{sign } (c(x,s)c(y,s))\, U(s) \left(e_{x_1\,y_1} \otimes e_{x_2\,y_2} \otimes \ldots \otimes e_{x_N\,y_N} \otimes \mathrm{I} \otimes \mathrm{I} \ldots \otimes \mathrm{I} \otimes \ldots\right) U(s)^* \\ &\quad = e_{x_{s^{-1}(1)}\,y_{s^{-1}(1)}} \otimes e_{x_{s^{-1}(2)}\,y_{s^{-1}(2)}} \otimes \ldots \otimes e_{x_{s^{-1}(N)}\,y_{s^{-1}(N)}} \otimes \mathrm{I} \otimes \mathrm{I} \ldots \otimes \mathrm{I} \otimes \ldots.\end{aligned}$$

If x, and y satisfies the condition:

$e_{x_i\,x_i}\mathcal{H} \subset \mathcal{H}_-$ if and only if, when $e_{y_i\,y_i}\mathcal{H} \subset \mathcal{H}_-$,

then, by definition cocycle c, we have $c(x,s) = c(y,s)$. Therefore,

$$\begin{aligned}&U(s) \left(e_{x_1\,y_1} \otimes e_{x_2\,y_2} \otimes \ldots \otimes e_{x_N\,y_N} \otimes \mathrm{I} \otimes \mathrm{I} \ldots\right) U(s)^* \\ &= e_{x_{s^{-1}(1)}\,y_{s^{-1}(1)}} \otimes e_{x_{s^{-1}(2)}\,y_{s^{-1}(2)}} \otimes \ldots \otimes e_{x_{s^{-1}(N)}\,y_{s^{-1}(N)}} \otimes \mathrm{I} \otimes \mathrm{I} \ldots.\end{aligned} \tag{44}$$

Hence, using properties (2)-(3) on the page 11, we obtain

$$\begin{aligned}&U(s) \left(\rho\left(\gamma_1\right) \otimes \rho\left(\gamma_2\right) \otimes \ldots \otimes \rho\left(\gamma_N\right) \otimes \ldots\right) U(s)^* \\ &= \rho\left(\gamma_{s^{-1}(1)}\right) \otimes \rho\left(\gamma_{s^{-1}(2)}\right) \otimes \ldots \otimes \rho\left(\gamma_{s^{-1}(N)}\right) \otimes \ldots\end{aligned} \tag{45}$$

for all $s \in \mathfrak{S}_\infty$, $\gamma_l \in \Gamma$.

Now we define the operators $\Pi_A^\rho(s)$, $(s \in \mathfrak{S}_\infty)$ and $\Pi_A^\rho(\gamma)$, $(\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty)$ on $\mathcal{H}_A^\rho$ as follows

$$\begin{aligned}\Pi_A^\rho(s)v = U(s)v, \quad v \in \mathcal{H}_A^\rho; \\ \Pi_A^\rho(\gamma)v = \left(\rho\left(\gamma_1\right) \otimes \rho\left(\gamma_2\right) \otimes \ldots\right) v.\end{aligned} \tag{46}$$

By (45), Π_A^ρ can be extended to the unitary representation of $\Gamma \wr \mathfrak{S}_\infty$.

The next proposition follows from the definition of Hilbert space $\mathcal{H}_A^\rho$ (see paragraph 2.2) and proposition 7.

Proposition 10. *Let I be the unit in $\mathcal{B}(\mathcal{H})^{\otimes\infty}$. Identify the elements of $\mathcal{B}(\mathcal{H})^{\otimes\infty}$ with the corresponding vectors in $\mathcal{H}_A^\rho$. Put $\psi_A^\rho(s\gamma) = (\Pi_A^\rho(s)\Pi_A^\rho(\gamma)I, I)$. Then ϕ_A^ρ is indecomposable $\mathfrak{S}_\infty$-central state on $\Gamma \wr \mathfrak{S}_\infty$ (see definitions 1 and 2).*

Let A_1, A_2 be the self-adjoint operators of the *trace class* (see [12]) from $\mathcal{B}(\mathcal{H})$ with the property $\mathrm{Tr}(|A_j|) \leq 1$, $(j = 1, 2)$, and let ρ_1, ρ_2 be the unitary representations of Γ: $\rho_i : \gamma \in \Gamma \mapsto \rho_i(\gamma) \in \mathcal{B}(\mathcal{H})$.

Proposition 11. *Let $\left(\mathcal{H}_i, A_i, \rho_i, \hat{\xi}_i\right)$, $i = 1, 2$ satisfy assumptions* (1)-(4) *(paragraph 2.1). Equality $\psi_{A_1}^{\rho_1} = \psi_{A_2}^{\rho_2}$ holds if and only if there exists isometry $\mathcal{U} : \mathcal{H}_1 \mapsto \mathcal{H}_2$ such that*

$$\hat{\xi}_2 = \mathcal{U}\hat{\xi}_1,\ A_2 = \mathcal{U}A_1\mathcal{U}^{-1} \text{ and } \rho_2(\gamma) = \mathcal{U}\rho_1(\gamma)\mathcal{U}^{-1} \text{ for all } \gamma \in \Gamma. \tag{47}$$

Proof. Assume (47) hold. It follows from (35) and proposition 10 that $\psi_{A_1}^{\rho_1} = \psi_{A_2}^{\rho_2}$.

Conversely, suppose that $\psi_{A_1}^{\rho_1} = \psi_{A_2}^{\rho_2}$.

Denote by ${\Pi_A^\rho}^0$ the restriction Π_A^ρ to subspace $[\Pi_A^\rho(\Gamma \wr \mathfrak{S}_\infty) I]$ generated by the vectors $\{\Pi_A^\rho(\Gamma \wr \mathfrak{S}_\infty) I\}$. Let $(l\,k)$ be the transposition interchanging l and k. According to the construction of representation Π_A^ρ and properties (i)-(ii) from paragraph 2.3, there exists operator

$$\mathcal{O}_l = w - \lim_{k\to\infty} \Pi_A^\rho((l\,k)) \tag{48}$$

and

$$\mathcal{O}_l(a_1 \otimes a_2 \otimes \ldots) = b_1 \otimes b_2 \otimes \ldots, \text{ where } b_k = \begin{cases} a_k, & \text{if } k \neq l, \\ Aa_k, & \text{if } k = l. \end{cases} \tag{49}$$

Let $\mathfrak{A}_l^{A\,\rho}$ be w^*-algebra in $\Pi_A^\rho(\Gamma \wr \mathfrak{S}_\infty)''$ generated by $\mathcal{O}_l$ and $\underbrace{\mathrm{I} \otimes \ldots \otimes \mathrm{I}}_{l-1} \otimes \rho(\gamma) \otimes \mathrm{I} \otimes \mathrm{I} \otimes \ldots$, $\gamma \in \Gamma$. Denote by $\mathcal{P}_0$ the orthogonal projection $\mathcal{H}_A^\rho$ onto $\left[\mathfrak{A}_l^{A\,\rho} I\right]$.

First we prove that w^*-algebra $\{A, \rho(\Gamma)\}'' \subset \mathcal{B}(\mathcal{H})$ generated by A and $\rho(\Gamma)$ is isomorphic to w^*-algebra $\mathfrak{A}_l^{A\,\rho}\mathcal{P}_0$. Namely, the map

$$\begin{gathered} \mathfrak{m}_l : A \mapsto \mathcal{O}_l\mathcal{P}_0, \\ \mathfrak{m}_l : \rho(\gamma) \mapsto \left(\underbrace{\mathrm{I} \otimes \ldots \otimes \mathrm{I}}_{l-1} \otimes \rho(\gamma) \otimes \mathrm{I} \otimes \mathrm{I} \otimes \ldots\right)\mathcal{P}_0 \end{gathered} \tag{50}$$

extends to an isomorphism of $\{A, \rho(\Gamma)\}''$ onto $\mathfrak{A}_l^{A\,\rho}\mathcal{P}_0$.

Using (49) and definition of Π_A^ρ, we can consider $\mathfrak{m}_l$ as the GNS-representation of $\{A, \rho(\Gamma)\}'' \subset \mathcal{B}(\mathcal{H})$ corresponding to ψ_k (see (35)). Thus

$\operatorname{Ker}\mathfrak{m}_l = \left\{a \in \{A, \rho(\Gamma)\}'' : \mathfrak{m}_l\left(a\right) = 0\right\}$ is weakly closed two-sided ideal. Therefore, there exists unique orthogonal projection e from the center of $\{A, \rho(\Gamma)\}''$ such that

$$\operatorname{Ker}\mathfrak{m}_l = e\left\{A, \rho(\Gamma)\right\}'' \text{ (see [14])}. \tag{51}$$

Let us prove that $e = 0$.

Denote by $c\left(\widetilde{P}\right)$ central support of orthogonal projection $\widetilde{P} \in \{A, \rho(\Gamma)\}'$: $\widetilde{P}\mathcal{H} = \widetilde{\mathcal{H}}$ (see property (2) from paragraph 2.1).

Let us first show that

$$e\, c\left(\widetilde{P}\right) = 0. \tag{52}$$

Conversely, suppose that $e\, c\left(\widetilde{P}\right) \neq 0$. Hence, since the map $\{A, \rho(\Gamma)\}''\ c\left(\widetilde{P}\right) \ni a \mapsto a\widetilde{P} \in \{A, \rho(\Gamma)\}''\,\widetilde{P}$ is isomorphism, we obtain $e\,\widetilde{P} \neq 0$. It follows from properties (1)-(3) (paragraph 2.1)) that $e\,\left(P_{]0,1]} + P_{[-1,0[}\right) \neq 0$. Thus, by (35), $\psi_l(e) \neq 0$. Therefore, $e \notin \operatorname{Ker}\mathfrak{m}_l$. This contradicts property (51).

Now, using (52) and property (2) (paragraph 2.1)), we have

$$e\,\left(I - c\left(\widetilde{P}\right)\right)\mathcal{H} \subseteq \mathcal{H}_{reg}. \tag{53}$$

Therefore, if $e\,\left(I - c\left(\widetilde{P}\right)\right) \neq 0$, then, using property (4) (paragraph 2.1), we obtain

$$e\,\left(I - c\left(\widetilde{P}\right)\right)\mathfrak{e}'_{l1}\hat{\xi} \neq 0. \tag{54}$$

Again, by (35), $\psi_l(e) \neq 0$ and $e \notin \operatorname{Ker}\mathfrak{m}_l$. It follows from (51) that

$$e\,\left(I - c\left(\widetilde{P}\right)\right) = 0. \tag{55}$$

Hence, using (52), we obtain

$$\operatorname{Ker}\mathfrak{m}_l = 0. \tag{56}$$

Now we suppose that $\phi_{A_1}^{\rho_1} = \phi_{A_2}^{\rho_2}$. Let $\mathcal{O}_l^{(1)}$ and $\mathcal{O}_l^{(2)}$ be the operators, which are defined by formula (48) for representations $\Pi_{A_1}^{\rho_1}$ and $\Pi_{A_2}^{\rho_2}$ respectively. If $\mathfrak{I}_l$ is the extension the map

$$\begin{gathered}\mathcal{O}_l^{(1)}\mathcal{P}_0 \mapsto \mathcal{O}_l^{(2)}\mathcal{P}_0,\\ \underbrace{\mathrm{I}\otimes\ldots\otimes\mathrm{I}}_{l-1}\otimes\rho_1(\gamma)\otimes\mathrm{I}\otimes\mathrm{I}\otimes\ldots \mapsto \underbrace{\mathrm{I}\otimes\ldots\otimes\mathrm{I}}_{l-1}\otimes\rho_2(\gamma)\otimes\mathrm{I}\otimes\mathrm{I}\otimes\ldots.\end{gathered}$$

by multiplication, then

$$(\mathfrak{I}_l(a)I, \mathfrak{I}_l(b)I) = (aI, bI) \text{ for all } a, b \in \mathfrak{A}_l^{A_1\,\rho_1}\mathcal{P}_0. \tag{57}$$

It follows from (56) that the map

$$\{A_1, \rho_1(\Gamma)\}'' \ni a \overset{\theta}{\mapsto} \mathfrak{m}_l^{-1} \circ \mathfrak{I}_l \circ \mathfrak{m}_l(a) \in \{A_2, \rho_2(\Gamma)\}'' \tag{58}$$

is an isomorphism. Since $\phi_{A_1}^{\rho_1} = \phi_{A_2}^{\rho_2}$, then, using definition of ϕ_A^ρ, in particular (35), obtain for all $v \in \{A_1, \rho_1(\Gamma)\}''$:

$$\begin{gathered} \operatorname{Tr}(v|A_1|) + (1 - \operatorname{Tr}(|A_1|)) \left(v\hat{\xi}_1, \hat{\xi}_1\right) \\ = \operatorname{Tr}(\theta(v)|A_2|) + (1 - \operatorname{Tr}(|A_2|)) \left(\theta(v)\hat{\xi}_2, \hat{\xi}_2\right). \end{gathered} \tag{59}$$

Without loss of generality we can assume that $\{A_1, \rho_1(\Gamma)\}'', \{A_2, \rho_2(\Gamma)\}'' \subset \mathcal{B}(\mathcal{H})$. Let $P_{[-1,0[}^{(i)}$, $P_{]0,1]}^{(i)}$ be the spectral projections of A_i ($i = 1, 2$). Put $P_\pm^{(i)} = P_{[-1,0[}^{(i)} + P_{]0,1]}^{(i)}$. It is clear $(\operatorname{Ker} A_i)^\perp = P_\pm^{(i)}\mathcal{H}$. Denote by $\widetilde{\mathcal{H}}_i$ subspace $\left[\{A_i, \rho_i(\Gamma)\}'' P_\pm^{(i)}\mathcal{H}_i\right]$. Let $\widetilde{P}_i$ be the orthogonal projection of $\mathcal{H}_i$ onto $\widetilde{\mathcal{H}}_i$. Put $P_{reg}^{(i)} = I - \widetilde{P}_i$. For $\alpha \in \operatorname{Spectrum} A_i$ denote by $P_\alpha^{(i)}$ the corresponding spectral projection.

Now, using properties of (A_i, ρ_i) (see paragraph 2.1), we have

$$\dim P_\alpha^{(i)}\mathcal{H} < \infty \quad \text{and} \quad P_\pm^{(i)} = \sum_{\alpha \in \operatorname{Spectrum} A_i : \alpha \neq 0} P_\alpha^{(i)}. \tag{60}$$

Therefore, there exists collection $\left\{c_j^{(i)}\right\}_{j=1}^N$ of pairwise orthogonal projections from the center of w^*-algebra $P_\pm^{(i)} \{A_i, \rho_i(\Gamma)\}'' P_\pm^{(i)}$ with properties

$$\begin{gathered} \theta\left(c_j^{(1)}\right) = c_j^{(2)} \text{ (see (58))} ; \quad \sum_{j=1}^N c_j^{(i)} = P_\pm^{(i)}; \\ c_j^{(i)} P_\pm^{(i)} \{A_i, \rho_i(\Gamma)\}'' P_\pm^{(i)} c_j^{(i)} \text{ is a factor of type } I_{n_j}. \end{gathered} \tag{61}$$

Fix matrix unit $\left\{f_{k\,l}^{(j)}\right\}_{k,l=1}^{n_j} \subset c_j^{(1)} P_\pm^{(1)} \{A_1, \rho_1(\Gamma)\}'' P_\pm^{(1)} c_j^{(1)}$, which is a linear basis in $c_j^{(1)} P_\pm^{(1)} \{A_1, \rho_1(\Gamma)\}'' P_\pm^{(1)} c_j^{(1)}$, minimal projections $\left\{f_{k\,k}^{(j)}\right\}_{k=1}^{n_j}$ satisfy condition

$$P_\alpha^{(1)} f_{k\,k}^{(j)} = f_{k\,k}^{(j)} P_\alpha^{(1)} \text{ for all } \alpha \in \operatorname{Spectrum} A_1;\ k, j \in \mathbb{N}. \tag{62}$$

Now, using (57), (58), (59) and definition of Π_A^ρ (see paragraphs 2.1,2.2, 2.3), we have

$$\operatorname{Tr}\left(f_{k\,k}^{(j)}\right) = \operatorname{Tr}\left(\theta\left(f_{k\,k}^{(j)}\right)\right) \text{ for all } k, j \in \mathbb{N}. \tag{63}$$

Therefore, there exists isometry $U : P_\pm^{(1)}\mathcal{H}_1 \mapsto P_\pm^{(1)}\mathcal{H}_2$ such that $UP_\pm^{(1)}\mathcal{H}_1 = P_\pm^{(1)}\mathcal{H}_2$ and

$$U f_{k\,k}^{(j)} U^{-1} = \theta\left(f_{k\,k}^{(j)}\right) \text{ for } k = 1, 2, \ldots n_j;\ j = 1, 2, \ldots, N. \tag{64}$$

Let $\mathcal{C}_i$ be the center of w^*-algebra $\{A_i, \rho_i(\Gamma)\}''$ and let $c\left(P_\pm^{(i)}\right) \in \mathcal{C}_i$ be the central support of $P_\pm^{(i)}$. It follows from this and (61) that there exist pairwise orthogonal projections $\left\{C_j^{(i)}\right\}_{j=1}^N \subset c\left(P_\pm^{(i)}\right) \cdot \mathcal{C}_i$ with the next properties

$$\begin{gathered} c_j^{(i)} = C_j^{(i)} \cdot P_\pm^{(i)}, \quad \sum_{j=1}^N C_j^{(i)} = c\left(P_\pm^{(i)}\right), \\ C_j^{(i)} \{A_i, \rho_i(\Gamma)\}'' \, C_j^{(i)} \text{ is a factor of type } I_{N_j}. \end{gathered} \tag{65}$$

In $C_j^{(1)} \{A_1, \rho_1(\Gamma)\}'' \, C_j^{(1)}$ there exists matrix unit $\left\{f_{k\,l}^{(j)}\right\}_{k,l=1}^{N_j}$ $(n_j \geq N_j)$. Now, applying (64), we obtain that

$$\widetilde{U} = \sum_{j=1}^N \sum_{k=1}^{N_j} \theta\left(f_{k1}^{(j)}\right) U f_{1k} \tag{66}$$

is an isometry of $c\left(P_\pm^{(1)}\right)\mathcal{H}_1$ onto $c\left(P_\pm^{(2)}\right)\mathcal{H}_2$. An easy computation shows that $\widetilde{U} f_{kl}^{(j)} \widetilde{U}^{-1} = \theta\left(f_{kl}^{(j)}\right)$ for $k, l = 1, 2, \ldots, N_j$; $j = 1, 2, \ldots, N$. Thus

$$\theta(a) = \widetilde{U} a \widetilde{U}^{-1} \text{ for all } a \in c\left(P_\pm^{(1)}\right) \{A_1, \rho_1(\Gamma)\}''. \tag{67}$$

Hence, using (59) and relations $\theta\left(|A_1|\right) = |A_2|$, $\theta\left(c\left(P_\pm^{(1)}\right)\right) = c\left(P_\pm^{(2)}\right)$, which follows from the definition of θ (see (58)), we have

$$\left(\left(I - c\left(P_\pm^{(2)}\right)\right)\theta(v)\hat{\xi}_2, \hat{\xi}_2\right) = \left(\left(I - c\left(P_\pm^{(1)}\right)\right) v\hat{\xi}_1, \hat{\xi}_1\right). \tag{68}$$

Since $\widetilde{P}_i \leq c\left(P_\pm^{(i)}\right)$, then

$$I - c\left(P_\pm^{(i)}\right) \leq P_{reg}^{(i)}, \quad i = 1, 2. \tag{69}$$

Denote by $\left\{\mathfrak{e}_{kl}^{(i)\prime},\ k, l \in \mathbb{N}\right\}$ $(i = 1, 2)$ the matrix unit from property (4) of paragraph 2.1. Now we define map V as follows

$$a\left(I - c\left(P_\pm^{(1)}\right)\right)\hat{\xi}_1 \overset{V}{\mapsto} \theta(a)\left(I - c\left(P_\pm^{(2)}\right)\right)\hat{\xi}_2, \text{ where } a \in \{A_1, \rho_1(\Gamma)\}''.$$

By (68) and (68), V extends to isometry V of $\left(I - c\left(P_\pm^{(1)}\right)\right)\mathfrak{e}_{11}^{(1)\prime}\mathcal{H}_1 \subset P_{reg}^{(1)}\mathcal{H}_1$ onto $\left(I - c\left(P_\pm^{(2)}\right)\right)\mathfrak{e}_{11}^{(1)\prime}\mathcal{H}_2 \subset P_{reg}^{(2)}\mathcal{H}_2$ and for all $a \in \{A_1, \rho_1(\Gamma)\}''$

$$V\left(I - c\left(P_\pm^{(1)}\right)\right) a\mathfrak{e}_{11}^{(1)\prime} V^{-1} = \left(I - c\left(P_\pm^{(2)}\right)\right)\theta(a)\mathfrak{e}_{11}^{(2)\prime}.$$

It follows from this that $\widetilde{V}=\sum\limits_{k=1}^{\infty}\mathfrak{e}_{k1}^{(2)\prime}V\left(I-c\left(P_{\pm}^{(1)}\right)\right)\mathfrak{e}_{1k}^{(1)\prime}$ is an isometry of $\left(I-c\left(P_{\pm}^{(1)}\right)\right)\mathcal{H}_1$ onto $\left(I-c\left(P_{\pm}^{(2)}\right)\right)\mathcal{H}_2$, satisfying the next relation

$$\widetilde{V}\left(I-c\left(P_{\pm}^{(1)}\right)\right)a\widetilde{V}^{-1}=\left(I-c\left(P_{\pm}^{(2)}\right)\right)\theta(a)\qquad\left(a\in\left\{A_1,\rho_1(\Gamma)\right\}''\right).$$

Hence, using (67), we obtain that $W=\widetilde{U}c\left(P_{\pm}^{(1)}\right)+\widetilde{V}\left(I-c\left(P_{\pm}^{(1)}\right)\right)$ is an isometry of $\mathcal{H}_1$ onto $\mathcal{H}_2$ and

$$WaW^{-1}=\theta(a)\text{ for all }\quad a\in\left\{A_1,\rho_1(\Gamma)\right\}''. \tag{70}$$

Now, on account of definition of θ and (59) one can easy to check that

$$\begin{gathered}W\hat{\xi}_1\perp\left[\left\{A_2,\rho_2(\Gamma)\right\}''P_{\pm}^{(2)}\mathcal{H}_2\right]=\widetilde{\mathcal{H}}_2\quad\text{ and}\\ \left(aW\hat{\xi}_1,W\hat{\xi}_1\right)=\left(a\hat{\xi}_2,\hat{\xi}_2\right)\text{ for all }\quad a\in\left\{A_2,\rho_2(\Gamma)\right\}''.\end{gathered}\tag{71}$$

Define linear map K by $K\left(v\right)=\begin{cases}a\hat{\xi}_2, & \text{if }\ v=aW\hat{\xi}_1\quad a\in a\in\left\{A_2,\rho_2(\Gamma)\right\}'',\\ 0, & \text{if }\ v\in\mathcal{H}_2\ominus\left[\left\{A_2,\rho_2(\Gamma)\right\}''\hat{\xi}_2\right].\end{cases}$
It follows from (71) that K extends to the partial isometry from $\left\{A_2,\rho_2(\Gamma)\right\}'$. Therefore, there exists unitary $\widetilde{K}\in\left\{A_2,\rho_2(\Gamma)\right\}'$ with the property: $\widetilde{K}v=Kv$ for all $v\in\left[\left\{A_2,\rho_2(\Gamma)\right\}''W\hat{\xi}_1\right]$. Thus $\mathcal{U}=\widetilde{K}W$ satisfies the conditions of proposition 11. □

2.4 The parameters of the states from paragraph 1.3. Here we follow the notation of paragraphs 1.3 and 2.1.

2.4.1 State φ_{sp}. Below we find parameters $\left(\mathcal{H},A,\widetilde{\mathcal{H}},\rho\right)$ from paragraph 2.1 such that $\varphi_{sp}=\psi_A^{\rho}$, where ψ_A^{ρ} defined in proposition 10.

Let $(\rho,\mathcal{H}_\varphi,\xi_\varphi)$ be GNS-representation of group Γ corresponding to φ, where $\varphi(\gamma)=(\rho(\gamma)\xi_\varphi,\xi_\varphi)$ for all $\gamma\in\Gamma$ and $\mathcal{H}_\varphi=[\rho\left(\Gamma\right)\xi_\varphi]$. An easy computation shows that $\mathcal{H}=\mathcal{H}_\varphi$, A acts by

$$A\xi=\left(\xi,\xi_\varphi\right)\xi_\varphi\quad(\xi\in\mathcal{H}),\tag{72}$$

and $\widetilde{\mathcal{H}}=\mathcal{H}$. It is clear $\mathcal{H}_{reg}=0$.

2.4.2 State φ_{reg}. As above $(\rho_\varphi,\mathcal{H}_\varphi,\xi_\varphi)$ is GNS-representation of Γ. If $\left(\rho_\varphi^{(k)},\mathcal{H}_\varphi^{(k)},\xi_\varphi^{(k)}\right)$ is k-th copy of $(\rho_\varphi,\mathcal{H}_\varphi,\xi_\varphi)$ then

$$\mathcal{H}=\mathcal{H}_{reg}=\bigoplus_{k=1}^{\infty}\left(\rho_\varphi^{(k)},\mathcal{H}_\varphi^{(k)},\xi_\varphi^{(k)}\right).$$

It is obvious, $A \equiv 0$. Now define $\mathfrak{e}'_{kl}$ by

$$\mathfrak{e}'_{kl}(\xi_1, \xi_2, \ldots) = \left(\underbrace{0, \ldots, 0}_{k-1}, \xi_l, 0, 0, \ldots \right).$$

Put $\rho = \bigoplus\limits_{k=1}^{\infty} \rho_\varphi^{(k)}$, $\hat{\xi} = (\xi_\varphi, 0, 0, \ldots)$. It is easy to check that $\varphi_{reg} = \psi_0^\rho$.

2.5 $\mathfrak{S}_\infty$-invariance of ψ_A^ρ. The next assertion follows from definition of ψ_A^ρ.

Proposition 12. *Let $s \in \mathfrak{S}_\infty$, $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_0^\infty$. If $s\gamma = \prod\limits_{p \in \mathbb{N}/s} s_p\gamma(p)$, where $s_p\gamma(p)$ is generalized cycle of $s\gamma$ (see (2)), then $\psi_A^\rho(s\gamma) = \prod\limits_{p \in \mathbb{N}/s} \psi_A^\rho(s_p\gamma(p))$. In particular, it follows from Proposition 7 that ψ_A^ρ is indecomposable state on $\Gamma \wr \mathfrak{S}_\infty$.*

Denote by $(n_1\ n_2\ \ldots\ n_k)$ cycle $\{n_1 \mapsto n_2 \mapsto \ldots \mapsto n_k \mapsto n_1\} \in \mathfrak{S}_\infty$. Suppose that $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty$ satisfies the condition: $\gamma_i = e$ for all $i \notin \{n_1, n_2, \ldots, n_k\}$. If $\operatorname{Tr}(|A|) = 1$, $c_k = (n_1\ n_2\ \ldots\ n_k)$ then, using (35), we have

$$\psi_A^\rho(c_k\gamma) = \operatorname{Tr}^{\otimes N}\left(U(c_k)\left(\rho(\gamma_1) \otimes \rho(\gamma_2) \otimes \ldots \otimes \rho(\gamma_N)\right) A^{\otimes N}\right) \tag{73}$$

for all $N \geq \max\{n_1, n_2, \ldots, n_k\}$, where $\operatorname{Tr}^{\otimes N}$ is the ordinary trace on $\mathcal{B}(\mathcal{H})^{\otimes N}$, $A^{\otimes N} = \underbrace{A \otimes \ldots \otimes A}_{N}$. The next lemma extends formula 73 on the general case.

Lemma 13. *If $k > 1$ then*

$$\psi_A^\rho(c_k\gamma) = \operatorname{Tr}^{\otimes N}\left(U((n_1\ n_2\ \ldots\ n_k))\left(\rho(\gamma_{n_1}) \otimes \rho(\gamma_{n_2}) \otimes \ldots \otimes \rho(\gamma_{n_k})\right) A^{\otimes k}\right).$$

Proof. Let $\widetilde{P}$ be an orthogonal projection on subspace $\widetilde{\mathcal{H}} = \mathcal{H}_+ \oplus \mathcal{H}_-$ (see paragraph 2.1). Put $E = E_1 \otimes E_2 \otimes \ldots \otimes E_N \otimes \ldots$, where $E_i = \begin{cases} \widetilde{P} + \mathfrak{e}'_{ii}, & \textit{if } i = n_j, \\ I_{\mathcal{H}}, & \textit{if } i \neq n_j \text{ for all } j \in \{1, 2, \ldots, k\}. \end{cases}$ Considering identical operator $I \in \mathcal{B}(\mathcal{H})$ as element of $\mathcal{H}_A^\rho$, we obtain from (35), (36), (37)

$$EI = I. \tag{74}$$

It follows from (44) that

$$\widetilde{E} = U(c_k)\, EU(c_k)^*\, E = \widetilde{E}_1 \otimes \widetilde{E}_2 \otimes \ldots \otimes \widetilde{E}_N \otimes \ldots, \tag{75}$$

where $\widetilde{E}_i = \begin{cases} \widetilde{P}, & \textit{if } i = n_j, \\ I_{\mathcal{H}}, & \textit{if } i \neq n_j \text{ for all } j \in \{1, 2, \ldots, k\}. \end{cases}$ By properties (1)-(4) from paragraph 2.1, using (46) and (44), we obtain

$$\Pi_A^\rho(\gamma)\, E = E\Pi_A^\rho(\gamma),\ \Pi_A^\rho(\gamma)\, \widetilde{E} = \widetilde{E}\Pi_A^\rho(\gamma). \tag{76}$$

Thus

$$\begin{aligned}\psi_A^\rho\left(c_k\gamma\right)&=\left(\Pi_A^\rho\left(c_k\right)\Pi_A^\rho(\gamma)I,I\right)\overset{(74)}{=}\left(\Pi_A^\rho\left(c_k\right)\Pi_A^\rho(\gamma)EI,EI\right)\\&=\left(\Pi_A^\rho\left(c_k\right)\Pi_A^\rho(\gamma)\Pi_A^\rho\left(c_k\right)^*\left[\Pi_A^\rho\left(c_k\right)E\Pi_A^\rho\left(c_k\right)^*\right]\Pi_A^\rho\left(c_k\right)I,EI\right)\\&\overset{(76)}{=}\left(\Pi_A^\rho\left(c_k\right)\Pi_A^\rho(\gamma)\Pi_A^\rho\left(c_k\right)^*\Pi_A^\rho\left(c_k\right)I,\left[\Pi_A^\rho\left(c_k\right)E\Pi_A^\rho\left(c_k\right)^*\right]EI\right)\\&\overset{(75)}{=}\left(\Pi_A^\rho\left(c_k\right)\Pi_A^\rho(\gamma)I,\widetilde{E}I\right)\overset{(75),(44)}{=}\left(\Pi_A^\rho\left(c_k\right)\Pi_A^\rho(\gamma)\widetilde{E}I,\widetilde{E}I\right).\end{aligned}\tag{77}$$

Hence, applying (35), (36), (37), obtain for $N\geq\max\{n_1,n_2,\ldots,n_k\}$ $\psi_A^\rho\left(c_k\gamma\right)={}_1\psi_N\left(\widetilde{E}U\left(c_k\right)\left(\rho\left(\gamma_1\right)\otimes\rho\left(\gamma_2\right)\otimes\ldots\otimes\rho\left(\gamma_N\right)\right)\widetilde{E}\right)$. Since $\widetilde{P}\perp\mathfrak{e}'_{kk}$ for all k, then ${}_1\psi_N\left(\widetilde{E}U\left(c_k\right)\left(\rho\left(\gamma_1\right)\otimes\rho\left(\gamma_2\right)\otimes\ldots\otimes\rho\left(\gamma_N\right)\right)\widetilde{E}\right)$
$=\mathrm{Tr}^{\otimes N}\left(U\left(\left(n_1\;n_2\;\ldots\;n_k\right)\right)\left(\rho\left(\gamma_{n_1}\right)\otimes\rho\left(\gamma_{n_2}\right)\otimes\ldots\otimes\rho\left(\gamma_{n_k}\right)\right)A^{\otimes k}\right)$. □

Remark 1. One should notice that in the case in which $c_k=1$,

$$\psi_A^\rho(\gamma)=\prod_{n=1}^{\infty}\left[\mathrm{Tr}\left(\rho\left(\gamma_n\right)|A|\right)+\left(1-\mathrm{Tr}\left(|A|\right)\right)\left(\rho\left(\gamma_n\right)\hat{\xi},\hat{\xi}\right)\right].\tag{78}$$

Hence, taking into account Proposition 12, Lemma 13 and (73), we obtain the next important property

$$\psi_A^\rho\left(sgs^{-1}\right)=\psi_A^\rho(g)\text{ for all }s\in\mathfrak{S}_\infty,g\in\Gamma\wr\mathfrak{S}_\infty.\tag{79}$$

3 KMS-condition for the $\mathfrak{S}_\infty$-central states.

3.1 KMS-condition for ψ_A^ρ. To the general definition of the KMS-condition we refer the reader to the book [15]. Here we introduce the definition of the KMS-condition for the indecomposable states only.

Definition 14. Let φ be an indecomposable state on the group G. Let $(\pi_\varphi,\mathcal{H}_\varphi,\xi_\varphi)$ be the corresponding GNS-construction, where ξ_φ is such that $\varphi(g)=(\pi_\varphi(g)\xi_\varphi,\xi_\varphi)$ for each $g\in G$. We say that φ satisfies the KMS-condition or φ is KMS-state, if ξ_φ is separating[2] for the w^*-algebra $\pi_\varphi(G)''$, generated by operators $\pi_\varphi(G)$.

The main result of this paragraph is the following:

Theorem 15. *Let $\left(A,\hat{\xi},\mathcal{H}_{reg},\mathfrak{e}'_{kl}\right)$ satisfy the conditions* (1)-(4) *from paragraph 2.1. State ψ_A^ρ satisfies the KMS-condition if and only if* $\operatorname{Ker}A=\mathcal{H}_{reg}$ *and $\hat{\xi}$ is cyclic and separating for the restriction $\rho_{11}=\rho\Big|_{\mathfrak{e}'_{11}\mathcal{H}_{reg}}$ of representation ρ to subspace $\mathfrak{e}'_{11}\mathcal{H}$.*

As a preliminary to the proof of the theorem, we will discuss two auxiliary lemmas.

[2]This means that for every $a\in\pi_\varphi(G)''$ the conditions $a\xi_\varphi=0$ and $a=0$ are equivalent.

Lemma 16. *Let $(\pi_{\psi_k}, H_{\psi_k}, \xi_{\psi_k})$ be GNS-representation of $\mathcal{B}(\mathcal{H})$ corresponding to state ψ_k (see (35)). Fix any $\epsilon > 0$ and denote by $P_{[\epsilon,1]}$ the spectral projection of $|A|$. Then for each $a \in \mathcal{B}(\mathcal{H})$ the map*

$$\mathfrak{R}_{P_{[\epsilon,1]}aP_{[\epsilon,1]}} : x \mapsto x \cdot P_{[\epsilon,1]}\, a\, P_{[\epsilon,1]}$$

may be extended by continuous to the bounded operator on H_{ψ_k} and $\left\|\mathfrak{R}_{P_{[\epsilon,1]}aP_{[\epsilon,1]}}\right\|_{H_{\psi_k}} \leq \frac{\|a\|}{\sqrt{\epsilon}}$.

Proof. Put $b = P_{[\epsilon,1]}aP_{[\epsilon,1]}$. Then

$$\begin{gathered}
(\mathfrak{R}_b x, \mathfrak{R}_b x)_{H_{\psi_k}} = \operatorname{Tr}(b|A|b^*x^*x) \leq \|b|A|b^*\| \operatorname{Tr}\left(P_{[\epsilon,1]}x^*x\right) \\
= \|b|A|b^*\| \cdot \operatorname{Tr}\left(|A| \cdot \left[\sum_{\lambda\in[\epsilon,1]\cap \operatorname{Spectrum}|A|} \lambda^{-1}P_\lambda\right] x^*x\right) \\
\leq \epsilon^{-1} \cdot \|b|A|b^*\| \cdot \operatorname{Tr}\left(|A|P_{[\epsilon,1]}x^*x\right) \leq \epsilon^{-1} \cdot \|b|A|b^*\| \cdot \operatorname{Tr}(|A|x^*x) \leq \\
\overset{35}{=} \epsilon^{-1} \cdot \|b|A|b^*\|\, \psi_k(x^*x) \leq \epsilon^{-1} \cdot \|b\|^2 (x^*x)_{H_{\psi_k}}.
\end{gathered}$$

□

Lemma 17. *Suppose that for $\left(A, \hat{\xi}, \mathcal{H}_{reg}, \mathfrak{e}'_{kl}\right)$ the conditions* (1)-(4) *from paragraph 2.1 hold. Denote by P_0 and P_{reg} the orthogonal projections onto* Ker A *and $\mathcal{H}_{reg}$ respectively. Let $[\Pi^\rho_A(\Gamma \wr \mathfrak{S}_\infty) I]$ be the subspace in $\mathcal{H}^\rho_A$ (see paragraphs 2.2, 2.3), generated by $\Pi^\rho_A(\Gamma \wr \mathfrak{S}_\infty) I$. For $m \in \{\rho(\Gamma)\}' \subset \mathcal{B}(\mathcal{H})$ define the linear map $\mathfrak{R}^{(k)}_m : \mathcal{B}(\mathcal{H})^{\otimes\infty} \mapsto \mathcal{B}(\mathcal{H})^{\otimes\infty}$ as follows*

$$\begin{aligned}
&\mathfrak{R}^{(k)}_m(a_1 \otimes \ldots \otimes a_k \otimes a_{k+1} \otimes \ldots) \\
&= a_1 \otimes \ldots \otimes a_k \cdot \mathfrak{e}'_{kk} \cdot m \cdot \mathfrak{e}'_{kk} \otimes a_{k+1} \otimes \ldots .
\end{aligned} \tag{80}$$

If $P_0 = P_{reg}$ then

- (i) $\mathfrak{R}^{(k)}_m(\Pi^\rho_A(\Gamma \wr \mathfrak{S}_\infty) I) \subset [\Pi^\rho_A(\Gamma \wr \mathfrak{S}_\infty) I]$;
- (ii) *the extension of $\left.\mathfrak{R}^{(k)}_m\right|_{\Pi^\rho_A(\Gamma\wr\mathfrak{S}_\infty)I}$ by continuous is bounded operator in* $[\Pi^\rho_A(\Gamma \wr \mathfrak{S}_\infty) I] \subset \mathcal{H}^\rho_A$.

Proof. To prove (i), it suffices to show that $\mathfrak{R}^{(k)}_m(I) \in [\Pi^\rho_A(\Gamma \wr \mathfrak{S}_\infty) I]$. Indeed, by property (4), for any $\epsilon > 0$ there exists $a_\epsilon = \sum_{g\in\Gamma_\epsilon} c_\gamma \rho(\gamma)$, where Γ_ϵ is a finite subset in Γ, satisfying

$$\left\|\mathfrak{e}'_{1k} m \mathfrak{e}'_{k1} \hat{\xi} - a_\epsilon \hat{\xi}\right\|_{\mathcal{H}} < \epsilon.$$

Hence, considering $\mathfrak{R}^{(k)}_m(I)$ and $a^{(k)}_\epsilon = \underbrace{I \otimes \ldots \otimes I}_{k-1} \otimes P_{reg} a_\epsilon P_{reg} \otimes I \otimes \ldots$ as the elements from $\mathcal{H}^\rho_A$, we have

$$\left\|\mathfrak{R}^{(k)}_m(I) - a^{(k)}_\epsilon\right\|_{\mathcal{H}^\rho_A} < \epsilon. \tag{81}$$

It follows from (48) and (49), that operator of the left multiplication on $\underbrace{I\otimes\ldots\otimes I}_{k-1}\otimes P_0\otimes I\otimes\ldots$ lies in $\Pi_A^\rho\left(\Gamma\wr\mathfrak{S}_\infty\right)''$. Hence, since $P_0=P_{reg}$, we get $a_\epsilon^{(k)}\in\Pi_A^\rho\left(\Gamma\wr\mathfrak{S}_\infty\right)''$. Therefore, using (81), we obtain $\mathfrak{R}_m^{(k)}(I)\in\left[\Pi_A^\rho\left(\Gamma\wr\mathfrak{S}_\infty\right)I\right]$.

Let us prove statement (ii). Put $\mathfrak{S}_\infty^{(k)}=\{s\in\mathfrak{S}_\infty : s(k)=k\}$. First, using (79), we observe that

$$\begin{gathered}(a_1b_1I,a_2b_2I)_{\mathcal{H}_A^\rho}=(a_1b_1b_2^*0I,a_2I)_{\mathcal{H}_A^\rho}\\ \text{for all } a_1,a_2\in\Pi_A^\rho\left(\Gamma\wr\mathfrak{S}_\infty\right)'' \text{ and } b_1,b_2\in\Pi_A^\rho\left(\mathfrak{S}_\infty\right)''.\end{gathered}\tag{82}$$

Denote be $\mathcal{L}_{P_0}^{(k)}$ operator of the left multiplication on $\underbrace{I\otimes\ldots\otimes I}_{k-1}\otimes P_0\otimes I\otimes\ldots$. By (48) and (49), $\mathcal{L}_{P_0}^{(k)}\in\Pi_A^\rho\left(\mathfrak{S}_\infty\right)''$. Therefore, $\left[\Pi_A^\rho\left(\Gamma\wr\mathfrak{S}_\infty\right)\left(I-\mathcal{L}_{P_0}^{(k)}\right)I\right]$, $\mathbf{H}_l=\left[\Pi_A^\rho\left((k\ l)\cdot\mathfrak{S}_\infty^{(k)}\right)\Pi_A^\rho\left(\Gamma_e^\infty\right)\mathcal{L}_{P_0}^{(k)}I\right]$ $(l\in\mathbb{N})$ are the subspaces in $\left[\Pi_A^\rho\left(\Gamma\wr\mathfrak{S}_\infty\right)I\right]$ and, according to (82), we have

$$\left[\Pi_A^\rho\left(\Gamma\wr\mathfrak{S}_\infty\right)\left(I-\mathcal{L}_{P_0}^{(k)}\right)I\right]\perp\mathbf{H}_l \text{ for all } l\in\mathbb{N}.\tag{83}$$

Now we prove that subspaces $\{\mathbf{H}_l\}_{l\in\mathbb{N}}$ are pairwise orthogonal. For convenience we assume that $k=1$. Denote by E_m the orthogonal projection on subspace $\mathbb{C}\mathfrak{e}'_{m1}\hat{\xi}\subset\mathcal{H}$ $(m\in\mathbb{N})$. Put $A_m=A+(I-\operatorname{Tr}|A|)\,E_m$, $\mathfrak{E}_m^{(i)\prime}=\underbrace{I\otimes\ldots\otimes I}_{i-1}\otimes\mathfrak{e}'_{mm}\otimes I\otimes\ldots$ and $E_m^{(i)}=\underbrace{I\otimes\ldots\otimes I}_{i-1}\otimes E_m\otimes I\otimes\ldots$. By definition,

$$E_m^{(i)}\mathfrak{E}_l^{(i)\prime}=\delta_{ml}E_m^{(i)}, \text{ where } \delta_{ml} \text{ is Kronecker's delta.}\tag{84}$$

It follows from the definition of A_m that for $s^{-1}(1)\neq 1$ and $n>s^{-1}(1)$

$$\mathfrak{E}_1^{\left(s^{-1}(1)\right)\prime}\cdot\bigotimes_{m=1}^n A_m=0.\tag{85}$$

Fix any $\widetilde{\gamma},\widehat{\gamma}\in\Gamma_e^\infty$, $s_1\in(1\ l_1)\,\mathfrak{S}_\infty^{(1)}$ and $s_2\in(1\ l_2)\,\mathfrak{S}_\infty^{(1)}$. Let us show that for $l_1\neq l_2$

$$\kappa=\left(\Pi_A^\rho\left(s_1\widetilde{\gamma}\right)\mathcal{L}_{P_0}^{(1)}I,\Pi_A^\rho\left(s_2\widehat{\gamma}\right)\mathcal{L}_{P_0}^{(1)}I\right)_{\mathcal{H}_A^\rho}=0.\tag{86}$$

Let $\operatorname{Tr}^{\otimes n}$ be the ordinary trace on w^*-factor $\mathcal{B}\left(\mathcal{H}\right)^{\otimes n}$. If $s=s_2^{-1}s_1$, $\gamma_m=\widehat{\gamma}_{s(m)}^{-1}\cdot\widetilde{\gamma}_m\in\Gamma$, $\gamma=(\gamma_1,\gamma_2,\ldots)$ and $n>\max\left\{\max\left\{i:\gamma_i\neq e\right\},\max\left\{i:s(i)\neq i\right\}\right\}$ then, using definition of Π_A^ρ (see (46)), we have

$$\kappa=\operatorname{Tr}^{\otimes n}\left(E_1^{(1)}\cdot U_n(s)\cdot\bigotimes_{m=1}^n\rho\left(\gamma_m\right)\cdot E_1^{(1)}\cdot\bigotimes_{m=1}^n A_m\right),\tag{87}$$

where $U_n(s)$ is defined in paragraph 2.3. Hence, applying property (4) from paragraph 2.1, (84) and (44), we obtain

$$\kappa = \mathrm{Tr}^{\otimes n}\left(E_1^{(1)} \cdot U_n(s)\ (U_n(s))^* \,\mathfrak{E}_1^{(1)\prime}\, U_n(s) \cdot \bigotimes_{m=1}^{n} \rho\left(\gamma_m\right) \cdot E_1^{(1)} \cdot \bigotimes_{m=1}^{n} A_m\right)$$

$$\overset{(44)}{=} \mathrm{Tr}^{\otimes n}\left(E_1^{(1)} \cdot U_n(s)\, \mathfrak{E}_1^{\left(s^{-1}(1)\right)\prime} \cdot \bigotimes_{m=1}^{n} \rho\left(\gamma_m\right) \cdot E_1^{(1)} \cdot \bigotimes_{m=1}^{n} A_m\right)$$

$$\overset{\mathrm{property}(4)}{=} \mathrm{Tr}^{\otimes n}\left(E_1^{(1)} \cdot U_n(s) \cdot \bigotimes_{m=1}^{n} \rho\left(\gamma_m\right) \cdot E_1^{(1)} \cdot \mathfrak{E}_1^{\left(s^{-1}(1)\right)\prime} \cdot \bigotimes_{m=1}^{n} A_m\right) \overset{(85)}{=} 0.$$

Therefore,

$$\mathbf{H}_l \perp \mathbf{H}_m \text{ for all } l \neq m. \tag{88}$$

As in the proof of (i), $\mathfrak{R}_m^{(1)}(I) = \mathfrak{e}'_{11} m \mathfrak{e}'_{11} \otimes I \otimes I \otimes \ldots$ lies in subspace $\left[\Pi_A^{\rho}\left(\Gamma_e^{\infty}\right) \mathfrak{L}_{P_0}^{(1)} I\right] \subset \mathbf{H}_1$. Therefore,

$$\Pi_A^{\rho}\left((1\ \ l) \cdot \mathfrak{S}_{\infty}^{(1)}\right) \Pi_A^{\rho}\left(\Gamma_e^{\infty}\right) \mathcal{L}_{P_0}^{(1)} \mathfrak{R}_m^{(1)}(I) \subset \mathbf{H}_l. \tag{89}$$

Further, using (44) and relation

$$\mathfrak{R}_m^{(1)} \Pi_A^{\rho}\left((1\ \ l) \cdot s\right) \Pi_A^{\rho}(\gamma) \mathcal{L}_{P_0}^{(1)}(I) \overset{(44)}{=} \mathcal{L}_{\mathfrak{e}'_{11} m \mathfrak{e}'_{11}}^{(l)} \Pi_A^{\rho}\left((1\ \ l) \cdot s\right) \Pi_A^{\rho}(\gamma) \mathcal{L}_{P_0}^{(1)}(I),$$

where $s \in \mathfrak{S}_{\infty}^{(1)}$, $\gamma \in \Gamma_e^{\infty}$, we obtain that $\mathfrak{R}_m^{(1)}$ is the bounded operator on $\mathbf{H}_l$ and $\left\|\mathfrak{R}_m^{(1)}\right\|_{\mathbf{H}_1} \leq \|\mathfrak{e}'_{11} m \mathfrak{e}'_{11}\|_{\mathcal{H}}$. Since, by (83) and (88),

$$\left[\Pi_A^{\rho}\left(\Gamma \wr \mathfrak{S}_{\infty}\right) I\right] = \left[\Pi_A^{\rho}\left(\Gamma \wr \mathfrak{S}_{\infty}\right)\left(I - \mathcal{L}_{P_0}^{(1)}\right) I\right] \bigoplus_{m=1}^{\infty} \mathbf{H}_m, \tag{90}$$

and $\left[\Pi_A^{\rho}\left(\Gamma \wr \mathfrak{S}_{\infty}\right)\left(I - \mathcal{L}_{P_0}^{(1)}\right) I\right] \subset \operatorname{Ker} \mathfrak{R}_m^{(1)}$, operator $\mathfrak{R}_m^{(1)}$ is bounded on subspace $\left[\Pi_A^{\rho}\left(\Gamma \wr \mathfrak{S}_{\infty}\right) I\right]$. □

The proof of Theorem 15. Let ${\Pi_A^{\rho}}^0$ be the restriction Π_A^{ρ} to subspace $\left[\Pi_A^{\rho}\left(\Gamma \wr \mathfrak{S}_{\infty}\right) I\right]$. Obvious, ${\Pi_A^{\rho}}^0$ and GNS-representation of $\Gamma \wr \mathfrak{S}_{\infty}$, corresponding to ψ_A^{ρ}, are naturally unitary equivalent. Let us prove that I is the cyclic vector for ${\Pi_A^{\rho}}^0\left(\Gamma \wr \mathfrak{S}_{\infty}\right)'$.

For any $n \in \mathbb{N}$ fix $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n, e, e, \ldots) \in \Gamma_e^{\infty}$ and $s \in \mathfrak{S}_n$. Put $\eta = \Pi_A^{\rho}(\gamma) I = \left(\bigotimes_{m=1}^{n} \rho\left(\gamma_m\right)\right) \otimes I \otimes I \otimes \ldots \in \left[{\Pi_A^{\rho}}^0\left(\Gamma_e^{\infty}\right) I\right] \subset \left[{\Pi_A^{\rho}}^0\left(\Gamma \wr \mathfrak{S}_{\infty}\right) I\right]$. If $P_{[\epsilon,1]}$ is the spectral projection of $|A|$ then, by (48), (49) and lemma 17 (i), for every $m'_j \in \rho(\Gamma)'$

$$a_{\epsilon} = \left(\bigotimes_{j=1}^{n}\left(P_{[\epsilon,1]} \rho\left(\gamma_j\right) P_{[\epsilon,1]} + \mathfrak{e}'_{jj} m'_j \mathfrak{e}'_{jj}\right)\right) \otimes I \otimes I \otimes \ldots \in \left[{\Pi_A^{\rho}}^0\left(\Gamma \wr \mathfrak{S}_{\infty}\right) I\right].$$

Since $\hat{\xi}$ is cyclic and separating for the restriction $\rho_{11} = \rho\Big|_{\mathfrak{e}'_{11}\mathcal{H}_{reg}}$ and $\operatorname{Ker} A = \mathcal{H}_{reg}$, then for any $\delta > 0$ there exist $\epsilon > 0$ and $\left\{m'_j\right\}_{j=1}^n \subset \rho(\Gamma)'$ such that

$$\|\Pi_A^\rho(\gamma)I - a_\epsilon\|_{\mathcal{H}_A^\rho} < \delta.$$

But, by lemmas 16-17, operator $\mathfrak{R}_{a_\epsilon}$ of right multiplication on a_ϵ lies in ${\Pi_A^\rho}^0 \left(\Gamma \wr \mathfrak{S}_\infty\right)'$. Therefore,

$$\Pi_A^\rho(\gamma)I \in \left[{\Pi_A^\rho}^0 \left(\Gamma \wr \mathfrak{S}_\infty\right)' I\right]. \tag{91}$$

Now we note that, by (79), the right multiplication on $U(s)$ defines the unitary operator $\mathfrak{R}_{U(s)} \in {\Pi_A^\rho}^0 \left(\Gamma \wr \mathfrak{S}_\infty\right)'$. It follows from (91) that $\Pi_A^\rho\left(\gamma\, s\right) I = \mathfrak{R}_{U(s)}\left(\Pi_A^\rho(\gamma)I\right) \in \left[{\Pi_A^\rho}^0 \left(\Gamma \wr \mathfrak{S}_\infty\right)' I\right]$. Therefore I is the cyclic vector for ${\Pi_A^\rho}^0 \left(\Gamma \wr \mathfrak{S}_\infty\right)'$.

Conversely, suppose that ψ_A^ρ is KMS-state on $\Gamma \wr \mathfrak{S}_\infty$. Define state $\widehat{\psi}_A^\rho \in {\Pi_A^\rho}^0 \left(\Gamma \wr \mathfrak{S}_\infty\right)''_*$ as follows

$$\widehat{\psi}_A^\rho(a) = (aI, I)_{\mathcal{H}_A^\rho}. \tag{92}$$

Then, by propositions 7 and 12, $\widehat{\psi}_A^\rho$ is faithful state. This means that for every $a \in {\Pi_A^\rho}^0 \left(\Gamma \wr \mathfrak{S}_\infty\right)''$ the conditions $\widehat{\psi}_A^\rho(a^*a) = 0$ and $a = 0$ are equivalent.

Let us prove that $\operatorname{Ker} A = \mathcal{H}_{reg}$. If $\mathcal{H}_{reg} \subsetneqq \operatorname{Ker} A$ then, by properties (1)-(4) from paragraph 2.1, there exists $\gamma \in \Gamma$ such that

$$\rho(\gamma)\left(P_{]0,1]} + P_{[-1,0[}\right) \neq \left(P_{]0,1]} + P_{[-1,0[}\right)\rho(\gamma). \tag{93}$$

It follows from this

$$Q = \left(\left(P_{]0,1]} + P_{[-1,0[}\right) \vee \rho(\gamma)\left(P_{]0,1]} + P_{[-1,0[}\right)\rho(\gamma)^*\right) - \left(P_{]0,1]} + P_{[-1,0[}\right) \neq 0.$$

Since $Q \in \mathfrak{A}$, where $\mathfrak{A}$ is defined in property (1) from paragraph 2.1, then, by (48)-(49), operator $\mathfrak{L}_Q^{(k)}$ of the left multiplication on $\left(\otimes_{m=1}^{k-1} I\right) \otimes Q \otimes I \otimes \ldots$ lies in ${\Pi_A^\rho}^0 \left(\Gamma \wr \mathfrak{S}_\infty\right)''$. Thus $\widehat{\psi}_A^\rho\left(\mathfrak{L}_Q^{(k)}\right) = \operatorname{Tr}\left(Q \cdot |A|\right) = 0$. But this contradicts the faithfulness of $\widehat{\psi}_A^\rho$.

Now we prove that $\hat{\xi}$ is cyclic and separating for the representation $\rho_{11} = \rho\Big|_{\mathfrak{e}'_{11}\mathcal{H}_{reg}}$. Denote by E_{11} the projection onto $\left[\rho_{11}(\Gamma)'\hat{\xi}\right]$ and suppose $\left[\rho_{11}(\Gamma)'\hat{\xi}\right] \subsetneqq \left[\rho_{11}(\Gamma)\hat{\xi}\right]$. It follows from this that

$$E_{11} \in \rho_{11}(\Gamma)'',\ F_{11} = \mathfrak{e}'_{11} - E_{11} \neq 0 \ \text{ and } \ F_{11}\hat{\xi} = 0. \tag{94}$$

Denote by P_{reg} the orthogonal projection onto $\mathcal{H}_{reg}$. Since $\operatorname{Ker} A = \mathcal{H}_{reg}$, then

$$P_{reg} \in \mathfrak{A} \ \text{ and } \ P_{reg} \cdot \rho\left(\Gamma\right)'' \cdot P_{reg} \subset \mathfrak{A}.$$

Hence, by properties (2) and (4) from paragraph 2.1, we obtain

$$F = \sum_{m=1}^{\infty} \mathfrak{e}'_{m1} \cdot F_{11} \cdot \mathfrak{e}'_{1m} \in P_{reg} \cdot \rho\,(\Gamma)'' .$$

Hence, using (48)-(49), we obtain that operator $\mathfrak{L}_F^{(k)}$ of the left multiplication on $\left(\otimes_{m=1}^{k-1} I\right) \otimes F \otimes I \otimes \ldots$ lies in $\Pi_A^{\rho\,0}\,(\Gamma \wr \mathfrak{S}_\infty)''$. It follows from this and (94) that $\widehat{\psi}_A^\rho \left(\mathfrak{L}_F^{(k)}\right) = 0$. □

4 The main result.

In this section we prove the main result of this paper:

Theorem 18. *Let φ be any indecomposable $\mathfrak{S}_\infty$-central state on the group $\Gamma \wr \mathfrak{S}_\infty$. Then there exist self-adjoint operator A of the trace class (see [12]) from $\mathcal{B}(\mathcal{H})$ and unitary representation ρ with the properties* (1)-(4) *(paragraph 2.1) such that $\varphi = \psi_A^\rho$ (see Proposition 10).*

We have divided the proof into a sequence of lemmas and propositions. First we introduce some new objects and notations.

4.1 Asymptotical transposition. Let $(\pi_\varphi, H_\varphi, \xi_\varphi)$ be GNS-representation of $\Gamma \wr \mathfrak{S}_\infty$ associated with φ, where $\varphi(g) = (\pi_\varphi(g)\xi_\varphi, \xi_\varphi)$ for all $g \in \Gamma \wr \mathfrak{S}_\infty$. In the sequel for convenience we denote group $\Gamma \wr \mathfrak{S}_\infty$ by G. Put

$$\begin{aligned} G_n(\infty) = \quad & \Big\{ s\gamma \in G \Big|\; s \in \mathfrak{S}_\infty, \gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty, \\ & s(l) = l \ \text{ and } \ \gamma_l = e \ \text{ for } \ l = 1, 2, \cdots, n \Big\}, \\ G_n = \quad & \left\{ s\gamma \in G \big|\; s(l) = l \ \text{ and } \ \gamma_l = e \ \text{ for all } \ l > n \right\}, \\ G^{(k)} = \quad & \left\{ s\gamma \in G \big|\; s(k) = k \ \text{ and } \ \gamma_k = e \right\}. \end{aligned}$$

It is clear that $G_0(\infty) = G$.

Proposition 19. *Let $(i\ j)$ denotes the transposition exchanging i and j. In the weak operator topology there exists $\lim\limits_{j\to\infty} \pi_\varphi\left((i\ j)\right)$.*

Proof. It is suffices to show that for any $g, h \in G$ there exists $\lim\limits_{j\to\infty} \left(\pi_\varphi\left((i\ j)\right) \pi_\varphi(g)\xi_\varphi, \pi_\varphi(g)\xi_\varphi\right)$. Find $N > i$ such that $g, h \in G_n$ for all $n \geq N$. Since φ is $\mathfrak{S}_\infty$-central, then

$$\begin{aligned} & \left(\pi_\varphi\left((i\ N)\right) \pi_\varphi(g)\xi_\varphi, \pi_\varphi(g)\xi_\varphi\right) \\ = & \left(\pi_\varphi\left((i\ j)\right) \pi_\varphi(g)\pi_\varphi((n\ N))\xi_\varphi, \pi_\varphi(g)\pi_\varphi((n\ N))\xi_\varphi\right) \\ & = \left(\pi_\varphi\left((i\ n)\right) \pi_\varphi(g)\xi_\varphi, \pi_\varphi(g)\xi_\varphi\right). \end{aligned}$$

Thus $\lim\limits_{j\to\infty} \left(\pi_\varphi\left((i\ j)\right) \pi_\varphi(g)\xi_\varphi, \pi_\varphi(g)\xi_\varphi\right) = \left(\pi_\varphi\left((i\ N)\right) \pi_\varphi(g)\xi_\varphi, \pi_\varphi(g)\xi_\varphi\right)$. □

We will call $\mathcal{O}_i = \lim\limits_{j\to\infty} \pi_\varphi\left((i\ j)\right)$ the *asymptotical* transposition.

4.2 The properties of the asymptotical transposition.

Lemma 20. *Let $g, h \in G^{(n)}$. Then for each $k \neq n$ the next relation holds:*

$$\left(\pi_\varphi\left(g \cdot (n\ k) \cdot h\right)\xi_\varphi, \xi_\varphi\right) = \left(\pi_\varphi(g)\mathcal{O}_k\pi_\varphi(h)\xi_\varphi, \xi_\varphi\right) \tag{95}$$

Proof. Fix $N \in \mathbb{N}$ such that $g, h \in G_N \cap G^{(n)}$. Then for each $m > N$ we have: $(n\ m) \cdot g = g \cdot (n\ m)$, $(n\ m) \cdot h = h \cdot (n\ m)$. Hence, by the $\mathfrak{S}_\infty$-centrality of φ, we obtain

$$\left(\pi_\varphi\left(g \cdot (n\ k) \cdot h\right)\xi_\varphi, \xi_\varphi\right) = \varphi\left(g \cdot (n\ k) \cdot h\right) = \varphi\left((n\ m) \cdot g \cdot (n\ k) \cdot h \cdot (n\ m)\right) = \\ \left(\pi_\varphi\left((n\ m) \cdot g(n\ k) \cdot h \cdot (n\ m)\right)\xi_\varphi, \xi_\varphi\right) = \left(\pi_\varphi\left(g \cdot (m\ k) \cdot h\right)\xi_\varphi, \xi_\varphi\right).$$

Approaching the limit as $m \to \infty$ we obtain the required assertion. □

Lemma 21. *The next relations hold true:*

(1) $\mathcal{O}_k\mathcal{O}_n = \mathcal{O}_n\mathcal{O}_k$ *for all* $k, n \in \mathbb{N}$*;*

(2) $\mathcal{O}_k\pi_\varphi(\gamma) = \pi_\varphi(\gamma)\mathcal{O}_k$ *for all* $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty$ *such that* $\gamma_k = e$*;*

(3) $\pi_\varphi(s)\mathcal{O}_k = \mathcal{O}_{s(k)}\pi_\varphi(s)$ *for all* $s \in \mathfrak{S}_\infty$*.*

The proof follows immediately from definition $\mathcal{O}_k$ (Proposition 19). The details are left the reader. □

We will use the notation $\mathfrak{A}_j$ for the W^*-algebra generated by the operators $\pi_\varphi(\gamma)$, where $\gamma = (e, \cdots, e, \gamma_j, e, \cdots)$ and $\mathcal{O}_j$. There is the natural isomorphism $\phi_{j,k}$ between $\mathfrak{A}_j$ and $\mathfrak{A}_k$ for any k and j:

$$\phi_{j,k} : \mathfrak{A}_k \to \mathfrak{A}_j,\ \phi_{j,k}(a) = \pi_\varphi\left((k\ j)\right)a\pi_\varphi\left((k\ j)\right). \tag{96}$$

Observe that $(\phi_{j,k}(a)\xi_\varphi, \xi_\varphi) = (a\xi_\varphi, \xi_\varphi)$ for all k, j and $a \in \mathfrak{A}_k$.

The next statement is the simple technical generalization of proposition 7.

Lemma 22. *Let $s = \prod\limits_{p \in \mathbb{N}/s} s_p$ be the decomposition of $s \in \mathfrak{S}_\infty$ into the product of cycles s_p, where $p \subset \mathbb{N}$ is the corresponding orbit. Fix any finite collection $\{U_j\}_{j=1}^N$ of the elements from $\pi_\varphi(G)''$. If $U_j \in \mathfrak{A}_j$ then*

$$\left(\pi_\varphi(s)\prod_j U_j\xi_\varphi, \xi_\varphi\right) = \prod_{p \in \mathbb{N}/s}\left(\pi_\varphi(s_p)\prod_{j \in p} U_j\xi_\varphi, \xi_\varphi\right). \tag{97}$$

Proposition 23. *Let $s_p \in \mathfrak{S}_\infty$ be the cyclic permutation on the set $p = \left\{k_1, k_2, \ldots, k_{|p|}\right\} \subset \mathbb{N}$, where $k_l = s^{1-l}(k_1)$. If $U_{k_i} \in \mathfrak{A}_{k_i}$ for all $k_i \in p$ then*

$$\begin{aligned} &\left(\pi_\varphi(s_p)U_{k_1}U_{k_2}\cdots U_{k_{|p|}}\xi_\varphi, \xi_\varphi\right) \\ &= \left(\phi_{k_{|p|}k_1}\left(U_{k_1}\right)\mathcal{O}_{k_{|p|}}\phi_{k_{|p|}k_2}\left(U_{k_2}\right)\mathcal{O}_{k_{|p|}}\cdots\mathcal{O}_{k_{|p|}}U_{k_{|p|}}\xi_\varphi, \xi_\varphi\right). \end{aligned} \tag{98}$$

Proof. For convenience we suppose that $p = \{1, 2, \ldots, n\}$ and

$$s_p(k) = \begin{cases} k-1, & \textit{if } k > 1 \\ n, & \textit{if } k = 1 \end{cases}.$$

Since $s_p = (1\ n)(2\ n)\cdots(n-1\ n)$, we obtain

$$\begin{aligned}
&(\pi_\varphi(s_p)\, U_1 U_2 \cdots U_n \xi_\varphi, \xi_\varphi) \\
&= (\pi_\varphi((1\ n)(2\ n)\cdots(n-2\ n))\, U_1 U_2 \cdots \pi_\varphi((n-1\ \ n))\, U_{n-1} U_n \xi_\varphi, \xi_\varphi) \\
&= (\pi_\varphi((1\ n)(2\ n)\cdots(n-2\ n))\, U_1 U_2 \cdots \phi_{n,n-1}(U_{n-1})\, \pi_\varphi((n-1\ \ n))\, U_n \xi_\varphi, \xi_\varphi)\,.
\end{aligned}$$

Hence, using $\mathfrak{S}_\infty$-invariance of φ and lemma 21, for any $N > n$ we have

$$\begin{aligned}
&(\pi_\varphi(s_p)\, U_1 U_2 \cdots U_n \xi_\varphi, \xi_\varphi) = (\pi_\varphi((n-1\ \ N) s_p (n-1\ \ N))\, U_1 U_2 \cdots U_n \xi_\varphi, \xi_\varphi) \\
&= (\pi_\varphi((1\ n)(2\ n)\cdots(n-2\ n))\, U_1 U_2 \cdots \phi_{n,n-1}(U_{n-1})\, \pi_\varphi((N\ \ n))\, U_n \xi_\varphi, \xi_\varphi)\,.
\end{aligned}$$

Approaching the limit as $N \to \infty$, we obtain

$$\begin{gathered}
(\pi_\varphi(s_p)\, U_1 U_2 \cdots U_n \xi_\varphi, \xi_\varphi) \\
= (\pi_\varphi((1\ n)(2\ n)\cdots(n-2\ n))\, U_1 U_2 \cdots U_{n-2} \phi_{n,n-1}(U_{n-1})\, \mathcal{O}_n U_n \xi_\varphi, \xi_\varphi)\,.
\end{gathered}$$

Since $\phi_{n,n-1}(U_{n-1})\, \mathcal{O}_n$, then, by the obvious induction, we have

$$\begin{gathered}
(\pi_\varphi(s_p)\, U_1 U_2 \cdots U_n \xi_\varphi, \xi_\varphi) \\
= (\phi_{n,1}(U_1)\, \mathcal{O}_n \phi_{n,2}(U_2)\, \mathcal{O}_n \cdots \phi_{n,n-2}(U_{n-2})\, \mathcal{O}_n \phi_{n,n-1}(U_{n-1})\, \mathcal{O}_n U_n \xi_\varphi, \xi_\varphi)\,.
\end{gathered}$$

□

The next statement is an analogue of Theorem 1 from [8].

Lemma 24. *Let $[a, b]$ belongs to $[-1, 0]$ or $[0, 1]$. with the property . Denote by $E^{(i)}_{[a,b]}$ the spectral projection of self-adjoint operator $\mathcal{O}_i$. If* $\min\{|a|, |b|\} > \varepsilon > 0$ *then* $\left(E^{(i)}_{[a,b]} \xi_\varphi, \xi_\varphi\right)^2 \geq \varepsilon \left(E^{(i)}_{[a,b]} \xi_\varphi, \xi_\varphi\right)$.

This result may be proved in much the same way as theorem 1 from [8]. For convenience we give below the full proof of lemma 24.

Proof. Using Lemma 20, we have

$$\begin{aligned}
&\left|\left(\pi_\varphi((i, i+1))\, E^{(i)}_{[a,b]} \xi_\varphi, E^{(i)}_{[a,b]} \xi_\varphi\right)\right| = \\
&\left|\left(\mathcal{O}_i E^{(i)}_{[a,b]} \xi_\varphi, E^{(i)}_{[a,b]} \xi_\varphi\right)\right| \geqslant \varepsilon \left|\left(E^{(i)}_{[a,b]} \xi_\varphi, \xi_\varphi\right)\right|.
\end{aligned} \tag{99}$$

Hence, applying (96) and lemma 21, we obtain

$$\begin{gathered}
E^{(i)}_{[a,b]} \pi_\varphi((i, i+1))\, E^{(i)}_{[a,b]} = E^{(i)}_{[a,b]} E^{(i+1)}_{[a,b]} \pi_\varphi((i, i+1)) = \\
E^{(i)}_{[a,b]} E^{(i+1)}_{[a,b]} \pi_\varphi((i, i+1))\, E^{(i)}_{[a,b]} E^{(i+1)}_{[a,b]}.
\end{gathered}$$

Therefore,

$$\begin{aligned}
&\left|\left(\pi_\varphi\left((i,i+1)\right)E^{(i)}_{[a,b]}\xi_\varphi, E^{(i)}_{[a,b]}\xi_\varphi\right)\right| \\
&= \left|\left(\pi_\varphi\left((i,i+1)\right)E^{(i)}_{[a,b]}E^{(i+1)}_{[a,b]}\xi_\varphi, E^{(i)}_{[a,b]}E^{(i+1)}_{[a,b]}\xi_\varphi\right)\right| \\
&\leq \left|\left(E^{(i)}_{[a,b]}E^{(i+1)}_{[a,b]}\xi_\varphi, \xi_\varphi\right)\right| \overset{(Lemma\ 22)}{=} \left(E^{(i)}_{[a,b]}E^{(i+1)}_{[a,b]}\xi_\varphi, \xi_\varphi\right)^2.
\end{aligned}$$

Hence, using (99), we obtain the statement of lemma 24. □

Let $P_0^{(i)}$ be the orthogonal projection on $\operatorname{Ker}\mathcal{O}_i$. Put $P_\pm^{(i)} = I - P_0^{(i)}$.

Lemma 25. *Vector ξ_φ is separating for w^*-algebra $P_\pm^{(j)}\mathfrak{A}_j P_\pm^{(j)}$.*

Proof. Let $V \in P_\pm^{(j)}\mathfrak{A}_j P_\pm^{(j)}$ and let $V\xi_\varphi = 0$. It suffices to show that

$$\left(\pi_\varphi(g)\,\xi_\varphi, \mathcal{O}_j V^* \pi_\varphi(h)\,\xi_\varphi\right) = 0 \text{ for all } g,h \in G. \tag{100}$$

First we note that, by $\mathfrak{S}_\infty$-invariance φ,

$$\pi_\varphi(s)\,V\pi_\varphi\left(s^{-1}\right)\xi_\varphi = 0 \text{ for all } s \in \mathfrak{S}_\infty. \tag{101}$$

Further, if $g \in G_N$ then for all $n > N$

$$\pi_\varphi\left((j\ n)\right)V^*\pi_\varphi\left((j\ n)\right)\pi_\varphi(g) = \pi_\varphi(g)\,\pi_\varphi\left((j\ n)\right)V^*\pi_\varphi\left((j\ n)\right).$$

Hence, using definition of $\mathcal{O}_j$ (see proposition 19),

$$\begin{aligned}
&\left(\pi_\varphi(g)\,\xi_\varphi, \mathcal{O}_j V^*\pi_\varphi(h)\,\xi_\varphi\right) = \lim_{n\to\infty}\left(\pi_\varphi(g)\,\xi_\varphi, \pi_\varphi\left((j\ n)\right)V^*\pi_\varphi(h)\,\xi_\varphi\right) \\
&= \lim_{n\to\infty}\left(\pi_\varphi\left((j\ n)\right)V\pi_\varphi\left((j\ n)\right)\xi_\varphi, \pi_\varphi\left(g^{-1}\right)\pi_\varphi\left((j\ n)\right)\pi_\varphi(h)\,\xi_\varphi\right) \overset{(101)}{=} 0.
\end{aligned}$$

Thus (100) is proved. □

The following statement is well known for the case of separating vector ξ_φ (see [8]). In our case it follows from lemmas 24 and 25.

Corollary 26. *There exist at most countable set of numbers α_i from $[-1,0)\cup(0,1]$ and a set of the pairwise orthogonal projections $\left\{P_{\alpha_i}^{(j)}\right\} \subset \mathfrak{A}_j$ such that*

$$\mathcal{O}_j = P_0^{(j)} + \sum_i \alpha_i P_{\alpha_i}^{(j)}. \tag{102}$$

Lemma 27. *Let $\alpha,\beta \in \operatorname{Spectrum}\mathcal{O}_j$. If $\alpha\beta < 0$ then $P_\alpha^{(j)}\mathfrak{A}_j P_\beta^{(j)} = 0$.*

Proof. By lemma 25, it suffices to show that

$$P_\alpha^{(j)} U P_\beta^{(j)} \xi_\varphi = 0 \text{ for all } U \in \mathfrak{A}_j. \tag{103}$$

First we note that

$$\left\| P_\alpha^{(j)} U P_\beta^{(j)} \xi_\varphi \right\|^2 = \left(P_\beta^{(j)} U^* P_\alpha^{(j)} U P_\beta^{(j)} \xi_\varphi, \xi_\varphi \right) = \frac{1}{\alpha} \left(P_\beta^{(j)} U^* P_\alpha^{(j)} \mathcal{O}_j U P_\beta^{(j)} \xi_\varphi, \xi_\varphi \right).$$

Hence, using proposition 23, we receive

$$\left\| P_\alpha^{(j)} U P_\beta^{(j)} \xi_\varphi \right\|^2 = \frac{1}{\alpha} \left(P_\beta^{(j)} U^* P_\alpha^{(j)} \pi_\varphi \left((j\ \ j+1) \right) P_\alpha^{(j)} U P_\beta^{(j)} \xi_\varphi, \xi_\varphi \right). \tag{104}$$

It follows from lemma 21 that

$$\begin{aligned}
\left\| P_\alpha^{(j)} U P_\beta^{(j)} \xi_\varphi \right\|^2 &= \frac{1}{\alpha} \left(P_\beta^{(j)} U^* P_\alpha^{(j)} \phi_{j+1,j} \left(P_\alpha^{(j)} U P_\beta^{(j)} \right) \pi_\varphi \left((j\ \ j+1) \right) \xi_\varphi, \xi_\varphi \right) \\
&= \frac{1}{\alpha} \left(\phi_{j+1,j} \left(P_\alpha^{(j)} U P_\beta^{(j)} \right) P_\beta^{(j)} U^* P_\alpha^{(j)} \pi_\varphi \left((j\ \ j+1) \right) \xi_\varphi, \xi_\varphi \right) \\
&= \frac{1}{\alpha} \left(\phi_{j+1,j} \left(P_\alpha^{(j)} U P_\beta^{(j)} \right) \pi_\varphi \left((j\ \ j+1) \right) \phi_{j+1,j} \left(P_\beta^{(j)} U^* P_\alpha^{(j)} \right) \xi_\varphi, \xi_\varphi \right) \\
&= \frac{1}{\alpha} \left(P_\alpha^{(j)} U P_\beta^{(j)} \pi_\varphi \left((j\ \ j+1) \right) P_\beta^{(j)} U^* P_\alpha^{(j)} \xi_\varphi, \xi_\varphi \right) \\
&\overset{\text{proposition 23}}{=} \frac{1}{\alpha} \left(P_\alpha^{(j)} U P_\beta^{(j)} \mathcal{O}_j P_\beta^{(j)} U^* P_\alpha^{(j)} \xi_\varphi, \xi_\varphi \right) = \frac{\beta}{\alpha} \left(P_\alpha^{(j)} U P_\beta^{(j)} U^* P_\alpha^{(j)} \xi_\varphi, \xi_\varphi \right) \leq 0.
\end{aligned}$$

Therefore, (103) holds true. □

The next assertion is an analogue of the theorem 2 from [8].

Lemma 28. *Let $\alpha \neq 0$ be the eigenvalue of operator $\mathcal{O}_j$ and let $P_\alpha^{(j)}$ be the corresponding spectral projection. Take any orthogonal projection $P \in P_\alpha^{(j)} \mathfrak{A}_j P_\alpha^{(j)}$ and put $\nu(P) = \left(P\xi_\varphi, \xi_\varphi \right) / |\alpha|$. Then $\nu(P) \in \mathbb{N} \cup \{0\}$.*

Proof. We use the arguments of Kerov, Olshanski, Vershik [1] and Okounkov [8]. Let $j = 1$.

First consider the case $\alpha > 0$. For $n \in \mathbb{N}$ put $\eta_n = \prod_{m=0}^{n-1} \phi_{1+m,1}(P) \xi_\varphi$. Let $s \in \mathfrak{S}_n$. In each orbit $p \in \mathbb{N}/s$ fix number $\mathfrak{s}(p)$. Since $\prod_{m=0}^{n-1} \phi_{1+m,1}(P)$ is an orthogonal projection and

$$\pi_\varphi(s) \cdot \prod_{m=0}^{n-1} \phi_{1+m,1}(P) = \prod_{m=0}^{n-1} \phi_{1+m,1}(P) \cdot \pi_\varphi(s),$$

then we have

$$
\begin{aligned}
(\pi_\varphi(s)\eta_n, \eta_n) &= \left(\pi_\varphi(s) \prod_{m=0}^{n-1} \phi_{1+m,1}(P)\xi_\varphi, \xi_\varphi\right) \\
&\overset{\text{lemma 22}}{=} \prod_{p\in\{\mathbb{N}/s: p\subset[1,n]\}} \left(\pi_\varphi(s_p) \prod_{k\in p} \phi_{k,j}(P)\xi_\varphi, \xi_\varphi\right) \\
&\overset{\text{prop 23}}{=} \prod_{p\in\{\mathbb{N}/s: p\subset[1,n]\}} \left(\phi_{\mathfrak{s}(p),1}(P)\cdot \mathcal{O}_{\mathfrak{s}(p)}\cdot \phi_{\mathfrak{s}(p),1}(P)\cdot \mathcal{O}_{\mathfrak{s}(p)}\cdots \mathcal{O}_{\mathfrak{s}(p)}\cdot \phi_{\mathfrak{s}(p),1}(P)\,\xi_\varphi, \xi_\varphi\right) \\
&= \prod_{p\in\{\mathbb{N}/s: p\subset[1,n]\}} \alpha^{|p|-1}\left(\phi_{\mathfrak{s}(p),1}(P)\,\xi_\varphi, \xi_\varphi\right) = \alpha^n \nu^{\,l(s)},
\end{aligned}
\tag{105}
$$

where $l(s)$ is the number of cycles in the decomposition of permutation s.

Now define orthogonal projection $Alt(n) \in \pi_\varphi(\mathfrak{S}_\infty)'' \subset \pi_\varphi(G)''$ by

$$Alt(n) = \frac{1}{n!} \sum_{s\in\mathfrak{S}_n} \operatorname{sign}(s)\ \pi_\varphi(s). \tag{106}$$

Using (105), we obtain:

$$(Alt(n)\eta_n, \eta_n) = \alpha^n \sum_{s\in\mathfrak{S}_n} \operatorname{sign}(s)\ \nu^{\,l(s)}. \tag{107}$$

In the same way as in [8], applying equality:

$$\sum_{s\in\mathfrak{S}_n} \operatorname{sign}(s)\ \nu^{\,l(s)} = \nu(\nu-1)\cdots(\nu-n+1),$$

we have

$$0 \le (\pi_\varphi(s)\eta_n, \eta_n) = \nu(\nu-1)\cdots(\nu-n+1). \tag{108}$$

Therefore, $\nu \in \mathbb{N}\cup\{0\}$.

The same proof remains for $\alpha < 0$. In above reasoning operator $Alt(n)$ it is necessary to replace by $Sym(n) = \frac{1}{n!} \sum\limits_{s\in\mathfrak{S}_n} \pi_\varphi(s)$. □

For $\alpha \in \operatorname{Spectrum} \mathcal{O}_j$ denote by $P_\alpha^{(j)}$ the corresponding spectral projection (see corollary 26). It follows from lemmas 25 and 28 that for $\alpha \neq 0$ w^*-algebra $P_\alpha^{(j)}\mathfrak{A}_j P_\alpha^{(j)}$ is finite dimensional. Therefore, there exists finite collection $\left\{P_{\alpha,i}^{(j)}\right\}_{i=1}^{n_\alpha} \subset P_\alpha^{(j)}\mathfrak{A}_j P_\alpha^{(j)}$ of the *pairwise orthogonal* projections with the properties:

$$
\begin{aligned}
&P_{\alpha,i}^{(j)}\xi_\varphi \neq 0 \ \text{ and } \ P_{\alpha,i}^{(j)} \ \text{ is minimal for all } i = 1, 2, \ldots,\ n_\alpha; \\
&\sum_{i=1}^{n_\alpha} P_{\alpha,i}^{(j)} = P_\alpha^{(j)}.
\end{aligned}
\tag{109}
$$

Proposition 29. *Let* $\mathcal{O}_j = P_0^{(j)} + \sum\limits_i \alpha_i P_{\alpha_i}^{(j)}$. *Put* $P_+^{(j)} = \sum\limits_{i:\alpha_i>0} P_{\alpha_i}^{(j)}$, $P_-^{(j)} = \sum\limits_{i:\alpha_i<0} P_{\alpha_i}^{(j)}$ *and* $P_\pm^{(j)} = P_+^{(j)} + P_-^{(j)}$. *Then for each* $U \in \mathfrak{A}_j$

$$P_\pm^{(j)} U P_0^{(j)} \xi_\varphi = 0.$$

Proof. It is suffice to prove that $P_\alpha^{(j)} U P_0^{(j)} \xi_\varphi = 0$ for all nonzero $\alpha \in \operatorname{Spectrum} \mathcal{O}_j$. But this fact follows from the next relations:

$$\begin{aligned}
\left(P_\alpha^{(j)} U P_0^{(j)} \xi_\varphi, P_\alpha^{(j)} U P_0^{(j)} \xi_\varphi\right) &= \left(P_0^{(j)} U^* P_\alpha^{(j)} U P_0^{(j)} \xi_\varphi, \xi_\varphi\right) \\
&= \frac{1}{\alpha}\left(P_0^{(j)} U^* \mathcal{O}_j P_\alpha^{(j)} U P_0^{(j)} \xi_\varphi, \xi_\varphi\right) \\
&\overset{lemma\, 20}{=} \frac{1}{\alpha}\left(P_0^{(j)} U^* \pi_\varphi\left((j\ \ j+1)\right) P_\alpha^{(j)} U P_0^{(j)} \xi_\varphi, \xi_\varphi\right) \\
&= \frac{1}{\alpha}\left(P_0^{(j)} U^* P_\alpha^{(j+1)} \cdot \phi_{j+1,j}(U) \cdot P_0^{(j+1)} \pi_\varphi\left((j\ \ j+1)\right) \xi_\varphi, \xi_\varphi\right) \\
&= \frac{1}{\alpha}\left(P_\alpha^{(j+1)} \cdot \phi_{j+1,j}(U) \cdot P_0^{(j+1)} \cdot P_0^{(j)} U^* \pi_\varphi\left((j\ \ j+1)\right) \xi_\varphi, \xi_\varphi\right) \\
&= \frac{1}{\alpha}\left(P_\alpha^{(j+1)} \cdot \phi_{j+1,j}(U) \cdot P_0^{(j+1)} \cdot \pi_\varphi\left((j\ \ j+1)\right) P_0^{(j+1)} \cdot \phi_{j+1,j}\left(U^*\right) \xi_\varphi, \xi_\varphi\right) \\
&\overset{lemma\, 20}{=} \frac{1}{\alpha}\left(P_\alpha^{(j+1)} \cdot \phi_{j+1,j}(U) \cdot P_0^{(j+1)} \cdot \mathcal{O}_{j+1} \cdot P_0^{(j+1)} \cdot \phi_{j+1,j}\left(U^*\right) \xi_\varphi, \xi_\varphi\right) = 0.
\end{aligned}$$

□

Put $\mathbb{H}_{reg}^{(j)} = \left[\mathfrak{A}_j P_0^{(j)} \xi_\varphi\right]$ and $\mathbb{H}_\pm^{(j)} = \left[\mathfrak{A}_j P_\pm^{(j)} \xi_\varphi\right]$. The next assertion follows from the previous proposition.

Corollary 30. (a) *Subspaces* $\mathbb{H}_{reg}^{(j)}$ *and* $\mathbb{H}_\pm^{(j)}$ *are orthogonal for each* $j \in \mathbb{N}$;

(b) *if* $\sum\limits_{\alpha \in \operatorname{Spectrum} \mathcal{O}_j : \alpha \neq 0} |\alpha| \cdot \nu\left(P_\alpha^{(j)}\right) = 1$ *(see lemma 28) then* $P_0^{(j)} \xi_\varphi = 0$.

Proof. Property (a) at once follows from proposition 29. To prove (b) we note that $1 = \left\|P_0^{(j)} \xi_\varphi\right\|^2 + \sum\limits_{\alpha \in \operatorname{Spectrum} \mathcal{O}_j : \alpha \neq 0} \left\|P_\alpha^{(j)} \xi_\varphi\right\|^2 \overset{\text{lemma } 28}{=} \left\|P_0^{(j)} \xi_\varphi\right\|^2$
$+ \sum\limits_{\alpha \in \operatorname{Spectrum} \mathcal{O}_j, \alpha \neq 0} \alpha \cdot \nu\left(P_\alpha^{(j)}\right)$. Therefore, $\left\|P_0^{(j)} \xi_\varphi\right\|^2 = 0$. □

Lemma 31. $\left(U \mathcal{O}_j V P_0^{(j)} \xi_\varphi, P_0^{(j)} \xi_\varphi\right) = 0$ *for all* $U, V \in \mathfrak{A}_j$.

The proof follows from the next relations:

$$\begin{aligned}
\left(U\mathcal{O}_j V P_0^{(j)}\xi_\varphi, P_0^{(j)}\xi_\varphi\right) &\overset{\text{lemma 20}}{=} \left(U\cdot\pi_\varphi\left((j\ \ j+1)\right)\cdot V P_0^{(j)}\xi_\varphi, P_0^{(j)}\xi_\varphi\right)\\
&= \left(P_0^{(j)}\cdot U\cdot\phi_{j+1,j}(V)\cdot P_0^{(j+1)}\cdot\pi_\varphi\left((j\ \ j+1)\right)\xi_\varphi,\xi_\varphi\right)\\
&= \left(\phi_{j+1,j}(V)\cdot P_0^{(j+1)}\cdot P_0^{(j)}\cdot U\cdot\pi_\varphi\left((j\ \ j+1)\right)\xi_\varphi,\xi_\varphi\right)\\
&= \left(\phi_{j+1,j}(V)\cdot P_0^{(j+1)}\cdot\pi_\varphi\left((j\ \ j+1)\right)\cdot P_0^{(j+1)}\cdot\phi_{j+1,j}(U)\xi_\varphi,\xi_\varphi\right)\\
&\overset{\text{lemma 20}}{=} \left(\phi_{j+1,j}(V)\cdot P_0^{(j+1)}\cdot\mathcal{O}_{j+1}\cdot P_0^{(j+1)}\cdot\phi_{j+1,j}(U)\xi_\varphi,\xi_\varphi\right) = 0.
\end{aligned}$$

□

Proposition 32. *Let* $\left\{P_{\alpha,i}^{(j)}\right\}_{i=1}^{n_\alpha}$ $(\alpha\in\{\mathrm{Spectrum}\,\mathcal{O}_j\}\setminus 0)$ *are the same as in (109). If* $P_{\alpha,i}^{(j)}\cdot P_{\beta,k}^{(j)}=0$ *then* $\left(P_{\alpha,i}^{(j)}\cdot U\cdot P_{\beta,k}^{(j)}\xi_\varphi,\xi_\varphi\right)=0$ *for all* $U\in\mathfrak{A}_j$.

Proof. The statement follows from the next relations:

$$\begin{aligned}
\left(P_{\alpha,i}^{(j)}\cdot U\cdot P_{\beta,k}^{(j)}\xi_\varphi,\xi_\varphi\right) &= \tfrac{1}{\alpha}\left(P_{\alpha,i}^{(j)}\cdot\mathcal{O}_j\cdot U\cdot P_{\beta,k}^{(j)}\xi_\varphi,\xi_\varphi\right)\\
&\overset{\text{lemma 20}}{=} \tfrac{1}{\alpha}\left(P_{\alpha,i}^{(j)}\cdot\pi_\varphi\left((j\ \ j+1)\right)\cdot U\cdot P_{\beta,k}^{(j)}\xi_\varphi,\xi_\varphi\right) =\\
&\tfrac{1}{\alpha}\left(P_{\alpha,i}^{(j)}\cdot\phi_{j+1,j}(U)\cdot P_{\beta,k}^{(j+1)}\cdot\pi_\varphi\left((j\ \ j+1)\right)\xi_\varphi,\xi_\varphi\right)\\
&= \tfrac{1}{\alpha}\left(\phi_{j+1,j}(U)\cdot P_{\beta,k}^{(j+1)}\cdot P_{\alpha,i}^{(j)}\cdot\pi_\varphi\left((j\ \ j+1)\right)\xi_\varphi,\xi_\varphi\right)\\
&= \tfrac{1}{\alpha}\left(\phi_{j+1,j}(U)\cdot P_{\beta,k}^{(j+1)}\cdot\pi_\varphi\left((j\ \ j+1)\right)\cdot P_{\alpha,i}^{(j+1)}\xi_\varphi,\xi_\varphi\right)\\
&\overset{\text{lemma 20}}{=} \tfrac{1}{\alpha}\left(\phi_{j+1,j}(U)\cdot P_{\beta,k}^{(j+1)}\cdot\mathcal{O}_{j+1}\cdot P_{\alpha,i}^{(j+1)}\xi_\varphi,\xi_\varphi\right)\\
&= \left(\phi_{j+1,j}(U)\cdot P_{\beta,k}^{(j+1)}\cdot P_{\alpha,i}^{(j+1)}\xi_\varphi,\xi_\varphi\right) = 0.
\end{aligned}$$

□

Now we give important

Corollary 33. *Let* $P_+^{(j)}$ *and* $P_-^{(j)}$ *are the same as in proposition 29. Then subspaces* $\left[\mathfrak{A}_j P_+^{(j)}\xi_\varphi\right]$ *and* $\left[\mathfrak{A}_j P_-^{(j)}\xi_\varphi\right]$ *are orthogonal.*

Proposition 34. *Let* $\left\{P_{\alpha,i}^{(j)}\right\}_{i=1}^{n_\alpha}$ $(\alpha\in\{\mathrm{Spectrum}\,\mathcal{O}_j\}\setminus 0)$ *are the same as in proposition 32. If there exists unitary* $U\in\mathfrak{A}_j$ *such that* $U\cdot P_{\alpha,i}^{(j)}\cdot U^*=P_{\beta,k}^{(j)}$ *then* $\frac{\left(P_{\alpha,i}^{(j)}\xi_\varphi,\xi_\varphi\right)}{|\alpha|}=\frac{\left(P_{\beta,k}^{(j)}\xi_\varphi,\xi_\varphi\right)}{|\beta|}$.

Proof. Let $\kappa_\alpha=\left(P_{\alpha,i}^{(j)}\xi_\varphi,\xi_\varphi\right)/|\alpha|$ and $\kappa_\beta=\left(P_{\beta,k}^{(j)}\xi_\varphi,\xi_\varphi\right)/|\beta|$. By lemma 28, $\kappa_\alpha,\kappa_\beta\in\mathbb{N}$. Suppose for the convenience that $j=1$. For any $n\in\mathbb{N}$, using (106)

and (107), we obtain

$$\begin{aligned}\left(Alt(n)\prod_{m=1}^{n}\phi_{m,1}\left(P_{\alpha,i}^{(1)}\right)\xi_{\varphi},\prod_{m=1}^{n}\phi_{m,1}\left(P_{\alpha,i}^{(1)}\right)\xi_{\varphi}\right)&=|\alpha|^{n}\prod_{m=0}^{n-1}\left(\kappa_{\alpha}-m\right);\\ \left(Alt(n)\prod_{m=1}^{n}\phi_{m,1}\left(P_{\beta,k}^{(1)}\right)\xi_{\varphi},\prod_{m=1}^{n}\phi_{m,1}\left(P_{\beta,k}^{(1)}\right)\xi_{\varphi}\right)&=|\beta|^{n}\prod_{m=0}^{n-1}\left(\kappa_{\beta}-m\right).\end{aligned}\tag{110}$$

This implies for $n=\kappa_\alpha+1$ that

$$\left(Alt(\kappa_\alpha+1)\prod_{m=1}^{\kappa_\alpha+1}\phi_{m,1}\left(P_{\alpha,i}^{(1)}\right)\xi_{\varphi},\prod_{m=1}^{\kappa_\alpha+1}\phi_{m,1}\left(P_{\alpha,i}^{(1)}\right)\xi_{\varphi}\right)=0.\tag{111}$$

Further, applying relation

$$Alt(n)\cdot\prod_{m=1}^{n}\phi_{m,1}(a)=\prod_{m=1}^{n}\phi_{m,1}(a)\cdot Alt(n)\ \text{ (for all } a\in\mathfrak{A}_1),$$

we get

$$\begin{aligned}0&\leq\left(Alt(\kappa_\alpha+1)\cdot\prod_{m=1}^{\kappa_\alpha+1}P_{\beta,k}^{(m)}\xi_\varphi,\prod_{m=1}^{\kappa_\alpha+1}P_{\beta,k}^{(m)}\xi_\varphi\right)\\ &=\left(Alt(\kappa_\alpha+1)\prod_{m=1}^{\kappa_\alpha+1}P_{\alpha,i}^{(m)}\phi_{m,1}\left(U^*\right)\xi_\varphi,\prod_{m=1}^{\kappa_\alpha+1}P_{\alpha,i}^{(m)}\phi_{m,1}\left(U^*\right)\xi_\varphi\right)\\ &=\left(Alt(\kappa_\alpha+1)\prod_{m=1}^{\kappa_\alpha+1}P_{\alpha,i}^{(m)}\phi_{m,1}\left(U^*\right)\xi_\varphi,\prod_{m=1}^{\kappa_\alpha+1}\phi_{m,1}\left(U^*\right)\xi_\varphi\right)\\ &=\tfrac{1}{\alpha^{\kappa_\alpha+1}}\left(Alt(\kappa_\alpha+1)\prod_{m=1}^{\kappa_\alpha+1}P_{\alpha,i}^{(m)}\mathcal{O}_m\phi_{m,1}\left(U^*\right)\xi_\varphi,\prod_{m=1}^{\kappa_\alpha+1}\phi_{m,1}\left(U^*\right)\xi_\varphi\right)\\ &\overset{\text{lemma 20}}{=}\tfrac{1}{\alpha^{\kappa_\alpha+1}}\left(Alt(\kappa_\alpha+1)\prod_{m=1}^{\kappa_\alpha+1}P_{\alpha,i}^{(m)}\pi_\varphi\left(\left(m\ \ \kappa_\alpha+1\right)\right)\phi_{m,1}\left(U^*\right)\xi_\varphi,\prod_{m=1}^{\kappa_\alpha+1}\phi_{m,1}\left(U^*\right)\xi_\varphi\right)\\ &=\tfrac{1}{\alpha^{\kappa_\alpha+1}}\left(Alt(\kappa_\alpha+1)\prod_{m=1}^{\kappa_\alpha+1}P_{\alpha,i}^{(m)}\phi_{m+\kappa_\alpha+1,1}\left(U^*\right)\pi_\varphi\left(\left(m\ \ \kappa_\alpha+1\right)\right)\xi_\varphi,\prod_{m=1}^{\kappa_\alpha+1}\phi_{m,1}\left(U^*\right)\xi_\varphi\right)\\ &=\tfrac{1}{\alpha^{\kappa_\alpha+1}}\left(Alt(\kappa_\alpha+1)\prod_{m=1}^{\kappa_\alpha+1}P_{\alpha,i}^{(m)}\pi_\varphi\left(\left(m\ \ \kappa_\alpha+1\right)\right)\xi_\varphi,\prod_{m=1}^{\kappa_\alpha+1}\phi_{m+\kappa_\alpha+1,1}\left(U^*\right)\phi_{m,1}\left(U^*\right)\xi_\varphi\right)\\ &\leq\tfrac{1}{|\alpha|^{\kappa_\alpha+1}}\left\|Alt(\kappa_\alpha+1)\prod_{m=1}^{\kappa_\alpha+1}P_{\alpha,i}^{(m)}\pi_\varphi\left(\left(m\ \ \kappa_\alpha+1\right)\right)\xi_\varphi\right\|\\ &=\tfrac{1}{|\alpha|^{\kappa_\alpha+1}}\left(Alt(\kappa_\alpha+1)\prod_{m=1}^{\kappa_\alpha+1}P_{\alpha,i}^{(m)}\pi_\varphi\left(\left(m\ \ \kappa_\alpha+1\right)\right)\xi_\varphi,\pi_\varphi\left(\left(m\ \ \kappa_\alpha+1\right)\right)\xi_\varphi\right)^{1/2}\\ &\overset{\mathfrak{S}_\infty\text{-centrality of }\varphi}{=}\tfrac{1}{|\alpha|^{\kappa_\alpha+1}}\left(Alt(\kappa_\alpha+1)\prod_{m=1}^{\kappa_\alpha+1}P_{\alpha,i}^{(m)}\xi_\varphi,\xi_\varphi\right)^{1/2}\overset{(111)}{=}0.\end{aligned}$$

Hence, applying (110), we have

$$\left(Alt(\kappa_\alpha+1)\cdot\prod_{m=1}^{\kappa_\alpha+1}P_{\beta,k}^{(m)}\xi_\varphi,\prod_{m=1}^{\kappa_\alpha+1}P_{\beta,k}^{(m)}\xi_\varphi\right)$$
$$=|\beta|^{\kappa_\alpha+1}\kappa_\beta\left(\kappa_\beta-1\right)\left(\kappa_\beta-2\right)\left(\kappa_\beta-\kappa_a\right)=0.$$

Therefore, $\kappa_\alpha\geq\kappa_\beta$. Similarly, $\kappa_\alpha\leq\kappa_\beta$. □

4.3 The proof of theorem 18. Now we will give the description of parameters (A,ρ) from paragraph 2.1, corresponding to φ.

First we describe the structure of w^*-algebra $\widetilde{P}_\pm^{(j)}\mathfrak{A}_j$, [3] where $\widetilde{P}_\pm^{(j)}$ is the orthogonal projection of $[\mathfrak{A}_j\xi_\varphi]$ onto $\left[\mathfrak{A}_jP_\pm^{(j)}\xi_\varphi\right]$ [4].

Let $\mathcal{C}_\pm^{(j)}$ be the center of $\widetilde{P}_\pm^{(j)}\mathfrak{A}_j$. Denote by $c(P)\in\mathcal{C}_\pm^{(j)}$ the central support of projection $P\in\widetilde{P}_\pm^{(j)}\mathfrak{A}_j$. Let us prove that

$$c\left(P_\pm^{(j)}\right)=\widetilde{P}_\pm^{(j)}. \tag{112}$$

Indeed, if $F=\widetilde{P}_\pm^{(j)}-c\left(P_\pm^{(j)}\right)$, then for all $B\in\mathfrak{A}_j$ we have $FBP_\pm^{(j)}\xi_\varphi=BFP_\pm^{(j)}\xi_\varphi=0$. Therefore, $F=0$.

Since for any nonzero $\alpha\in\{\mathrm{Spectrum}\,\mathcal{O}_j\}\setminus 0$ in $P_\alpha^{(j)}\mathfrak{A}_jP_\alpha^{(j)}$ there exists finite collection $\left\{P_{\alpha,i}^{(j)}\right\}_{i=1}^{n_\alpha}$ of the *minimal* projections with properties (109), then w^*-algebra $P_\pm^{(j)}\mathfrak{A}_jP_\pm^{(j)}$ is $*$-isomorphic to the direct sum of full matrix algebras. Thus, using (112), we find the collection $\{F_m\}_{m=1}^N$ of pairwise orthogonal projections from $\mathcal{C}_\pm^{(j)}$ such that $F_m\cdot\widetilde{P}_\pm^{(j)}\mathfrak{A}_j\cdot F_m$ is a factor of the type I_{k_m}. Denote $F_m\cdot\widetilde{P}_\pm^{(j)}\mathfrak{A}_j\cdot F_m$ by $\mathcal{M}_{k_m}$. That is $P_\pm^{(j)}\mathfrak{A}_jP_\pm^{(j)}$ is isomorphic to $\mathcal{M}_{k_1}\oplus\mathcal{M}_{k_2}\oplus\ldots$. Let $\left\{e_{pq}^{(m)}\right\}_{p,q=1}^{k_m}$ be the matrix unit of $\mathcal{M}_{k_m}$. Without loss of the generality we suppose that for certain $l_m\leq k_m$

$$\begin{gathered}\bigcup_m\left\{e_{pp}^{(m)}\right\}_{p=1}^{l_m}\subset\bigcup_{\alpha\in\mathrm{Spectrum}\,\mathcal{O}_j,\,\alpha\neq0}\left\{P_{\alpha,i}^{(j)}\right\}_{i=1}^{n_\alpha}\quad\text{and}\\ \left\{\bigcup_m\left\{e_{pp}^{(m)}\right\}_{p=l_m+1}^{k_m}\right\}\bigcap\left\{\bigcup_{\alpha\in\mathrm{Spectrum}\,\mathcal{O}_j,\,\alpha\neq0}\left\{P_{\alpha,i}^{(j)}\right\}_{i=1}^{n_\alpha}\right\}=\emptyset.\end{gathered}\tag{113}$$

By lemmas 25, 28 and propositions 32, 34, minimal projections $\bigcup_m\left\{e_{pp}^{(m)}\right\}_{p=1}^{l_m}$ satisfy the next conditions

[3] see page 26 for the definition of $\mathfrak{A}_j$

[4] $P_\pm^{(j)}$ is defined in proposition 29

- (a) if $e_{pp}^{(m)} \cdot \mathcal{O}_j = \alpha_p \cdot e_{pp}^{(m)}$, where $\alpha_p \in \text{Spectrum} \setminus 0$, then there exists natural q_m such that $\frac{\left(e_{pp}^{(m)}\xi_\varphi,\xi_\varphi\right)}{|\alpha_p|} = q_m$ for all $p = 1, 2, \ldots, l_m$;
- (b) if $p \neq q$ then $\left(e_{pq}^{(m)}\xi_\varphi, \xi_\varphi\right) = 0$ for all $p, q = 1, 2, \ldots, l_m$; $m = 1, 2, \ldots, N$.

Further, using (113), for $p > l_m$ we have

$$e_{pp}^{(m)} \cdot P_0^{(j)} = e_{pp}^{(m)}.$$

It follows from this and proposition 29 that

$$\left(e_{pq}^{(m)}\xi_\varphi, \xi_\varphi\right) = 0 \ \text{ for } \quad p = 1, 2, \ldots, l_m; \ \ q = l_m + 1, l_m + 2, \ldots, k_m. \tag{114}$$

Let us prove that

$$\left(e_{pq}^{(m)}\xi_\varphi, \xi_\varphi\right) = 0 \ \text{ for } \quad p, q = l_m + 1, l_m + 2, \ldots, k_m. \tag{115}$$

For this it suffices to prove the next equality:

$$\left(e_{pp}^{(m)}\xi_\varphi, \xi_\varphi\right) = 0 \ \text{ for } \quad p, q = l_m + 1, l_m + 2, \ldots, k_m. \tag{116}$$

Fix $p > l_m$. Applying proposition 23, we have

$$\begin{aligned}
\left(e_{pp}^{(m)}\xi_\varphi, \xi_\varphi\right) &= \tfrac{1}{\alpha_1}\left(e_{p1}^{(m)} \cdot \mathcal{O}_j \cdot e_{1p}^{(m)}\xi_\varphi, \xi_\varphi\right) \\
&\overset{\text{proposition 23}}{=} \tfrac{1}{\alpha_1}\left(\pi_\varphi\left((j\ \ j+1)\right) \cdot \phi_{j+1,j}\left(e_{p1}^{(m)}\right) \cdot e_{1p}^{(m)}\xi_\varphi, \xi_\varphi\right) \\
&= \tfrac{1}{\alpha_1}\left(\pi_\varphi\left((j\ \ j+1)\right) \cdot e_{1p}^{(m)} \cdot \phi_{j+1,j}\left(e_{p1}^{(m)}\right)\xi_\varphi, \xi_\varphi\right) \\
&= \tfrac{1}{\alpha_1}\left(\phi_{j+1,j}\left(e_{1p}^{(m)}\right) \cdot \pi_\varphi\left((j\ \ j+1)\right) \cdot \phi_{j+1,j}\left(e_{p1}^{(m)}\right)\xi_\varphi, \xi_\varphi\right) \\
&= \tfrac{1}{\alpha_1}\left(\pi_\varphi\left((j\ \ n)\right) \cdot \phi_{j+1,j}\left(e_{1p}^{(m)}\right) \cdot \pi_\varphi\left((j\ \ j+1)\right) \cdot \phi_{j+1,j}\left(e_{p1}^{(m)}\right) \cdot \pi_\varphi\left((j\ \ n)\right)\xi_\varphi, \xi_\varphi\right) \\
&= \tfrac{1}{\alpha_1}\left(\phi_{j+1,j}\left(e_{1p}^{(m)}\right) \cdot \pi_\varphi\left((j+1\ \ n)\right) \cdot \phi_{j+1,j}\left(e_{p1}^{(m)}\right)\xi_\varphi, \xi_\varphi\right) \\
&= \lim_{n\to\infty} \tfrac{1}{\alpha_1}\left(\phi_{j+1,j}\left(e_{1p}^{(m)}\right) \cdot \pi_\varphi\left((j+1\ \ n)\right) \cdot \phi_{j+1,j}\left(e_{p1}^{(m)}\right)\xi_\varphi, \xi_\varphi\right) \\
&= \tfrac{1}{\alpha_1}\left(\phi_{j+1,j}\left(e_{1p}^{(m)}\right) \cdot \mathcal{O}_{j+1} \cdot \phi_{j+1,j}\left(e_{p1}^{(m)}\right)\xi_\varphi, \xi_\varphi\right) \\
&= \tfrac{1}{\alpha_1}\left(e_{1p}^{(m)} \cdot \mathcal{O}_j \cdot e_{p1}^{(m)}\xi_\varphi, \xi_\varphi\right) \overset{(113)}{=} 0.
\end{aligned}$$

Thus (116) and (115) are proved.

Define $\widehat{\varphi} \in \pi_\varphi(G)''_*$ by $\widehat{\varphi}(a) = (a\xi_\varphi, \xi_\varphi)$. Denote by $\mathbb{M}_{q_m}$ the algebra of all complex matrices and put $\mathcal{N}_m = \mathcal{M}_{k_m} \otimes \mathbb{M}_{q_m}$, $A^{(m)} = \sum\limits_{p=1}^{l_m} \alpha_p \cdot$

$e_{pp}^{(m)} \in \mathcal{M}_{k_m}$ (see property (a) and (113)). Consider w^*-algebra $\widetilde{\mathfrak{A}}_j = \left(\bigoplus\limits_{m=1}^{N} F_m \mathfrak{A}_j F_m \otimes \mathbb{M}_{q_m}\right) \bigoplus \left(I - \widetilde{P}_{\pm}^{(j)}\right) \mathfrak{A}_j$. Observe that there exists the natural embedding

$$\mathfrak{A}_j \ni a \overset{\mathfrak{i}}{\mapsto} \sum_{m=1}^{N} (F_m a F_m \otimes I) + \left(I - \widetilde{P}_{\pm}^{(j)}\right) a \in \widetilde{\mathfrak{A}}_j. \tag{117}$$

Now, using properties (a)-(b), (114) and (115), we have for all $a \in \mathfrak{A}_j$

$$\widehat{\varphi}(a) = \sum_{m=1}^{N} \mathrm{Tr}_m \left(a \left| A^{(m)} \right| \otimes I\right) + \left(a \left(I - \widetilde{P}_{\pm}^{(j)}\right) \xi_\varphi, \xi_\varphi\right), \tag{118}$$

where Tr_m is ordinary trace[5] on $\mathcal{N}_m$.

Now we define parameters $\left\{\mathcal{H}, A, \rho, \hat{\xi}\right\}$ from paragraph 2.1 such that

$$\varphi = \psi_A^\rho \ \text{(see proposition 10)}. \tag{119}$$

For this purpose we fix in each $\mathcal{N}_m = \mathcal{M}_{k_m} \otimes \mathbb{M}_{q_m}$ minimal projection e_m. Define state f on $\widetilde{\mathfrak{A}}_j$ by

$$f(\widetilde{a}) = \sum_{m=1}^{N} \mathrm{Tr}_m (e_m \widetilde{a} e_m) \quad \left(\widetilde{a} \in \widetilde{\mathfrak{A}}_j\right). \tag{120}$$

Let $(R_f, \mathcal{H}_f, \xi_f)$ be the corresponding GNS-representation of $\widetilde{\mathfrak{A}}_j$. Now we define $\mathcal{H}$ by

$$\mathcal{H} = \mathcal{H}_f \oplus \left[\left(I - \widetilde{P}_{\pm}^{(1)}\right) \mathfrak{A}_1 \xi_\varphi\right] \oplus \left[\left(I - \widetilde{P}_{\pm}^{(2)}\right) \mathfrak{A}_2 \xi_\varphi\right] \oplus \ldots. \tag{121}$$

Representation ρ acts on $\eta_p \in \left[\left(I - \widetilde{P}_{\pm}^{(p)}\right) \mathfrak{A}_p \xi_\varphi\right]$ as follows

$$\rho(\gamma)\eta_p = \pi_\varphi\left(\left(e, \ldots, \overset{p-th}{\gamma}, e, \ldots\right)\right)\eta_p. \tag{122}$$

If $\eta \in \mathcal{H}_f$ then

$$\rho(\gamma)\eta = R_f \circ \mathfrak{i}\left(\pi_\varphi\left(\left(e, \ldots, \overset{j-th}{\gamma}, e, \ldots\right)\right)\right)\eta. \tag{123}$$

Operator A is defined by

$$A\eta = \begin{cases} R_f \circ \mathfrak{i}\left(\sum\limits_{m=1}^{N} A^{(m)}\right)\eta, & \text{if } \eta \in \mathcal{H}_f, \\ 0, & \text{if } \eta \in \left[\left(I - \widetilde{P}_{\pm}^{(p)}\right) \mathfrak{A}_p \xi_\varphi\right]. \end{cases} \tag{124}$$

[5]If e is minimal projection from $\mathcal{N}_m$ then $\mathrm{Tr}_m(e) = 1$.

In the case $\sum\limits_{\alpha\in\mathrm{Spectrum}\,\mathcal{O}_j,\,\alpha\neq 0}|\alpha|\nu\left(P_\alpha^{(j)}\right) = \sum\limits_{m=1}^{N}\sum\limits_{p=1}^{k_m}|\alpha_p| < 1$ (see corollary 30 and property (a)) vector $\hat{\xi}$ is defined by

$$\hat{\xi} = \frac{\left(I - \widetilde{P}_\pm^{(1)}\right)\xi_\varphi}{\left\|\left(I - \widetilde{P}_\pm^{(1)}\right)\xi_\varphi\right\|}. \tag{125}$$

Now it follows from (118) that for $a \in \mathfrak{A}_j$

$$\begin{aligned} &\widehat{\varphi}(a) = \mathrm{Tr}\left(R_f\left(\mathfrak{i}(a)\right)\cdot|A|\right) \\ &+\left\|\left(I - \widetilde{P}_\pm^{(1)}\right)\xi_\varphi\right\|\left(\pi_\varphi\left((1\ j)\right)\cdot a\cdot\pi_\varphi\left((1\ j)\right)\hat{\xi},\hat{\xi}\right). \end{aligned} \tag{126}$$

Hence, applying lemma 22, proposition 23 and definition of ψ_A^ρ, we can to receive equality (119). In particular, lemma 27 implies property (3) from paragraph 2.1. $\square$

References

[1] S.Kerov, G.Olshanski, A.Vershik, *Harmonic analysis on the infinite symmetric group*, RT-0312270.

[2] G.Olshanski, *An introduction to harmonic analysis on the infinite symmetric group*, RT-0311369.

[3] G.Olshanski, *Unitary representations of* $(G,K)-$*pairs connected with the infinite symmetric group* $S(\infty)$, Algebra i Analiz **1** (1989), no. 4, 178-209 (Russian); English translation in Leningrad Math. J. **1** (1990), no. 4, 983-1014.

[4] A.M. Vershik and S.V. Kerov, *Asymptotic theory of characters of the infinite symmetrc group*, Funct. Anal. Appl., **15** (1981), 246-255.

[5] A.M. Vershik and S.V. Kerov, *Characters and factor representations of the infinite symmetric group,* Soviet Math. Dokl., **23** (1981), no. 2, 389–392.

[6] R.Boyer, *Character theory of infinite wreath products*, Inter. Journal of Mathematics and Math. Sciences, **9** (2005),1365-1379.

[7] A.Okounkov, *The Thoma theorem and representation of the infinite bisymmetric group*, Funct. Anal. Appl. **28** (1994), no. 2, 100-107.

[8] A.Okounkov, *On the representation of the infinite symmetric group*, RT-9803037.

[9] Dudko, A.; Nessonov, N. *A description of characters on the infinite wreath product*, arXiv: math.RT/0510597, 33pp.

[10] Dudko A. V. , Nessonov N. I. *A description of characters on the infinite wreath product*, Methods of functional analysis and topology, Volume 13 (2007), Number 4, 301-317.

[11] G.Olshanski and A.Vershik, *Ergodic unitary invariant measures on the space of infinite Hermitian matrices*, Contemporary Mathematical Physics (R.L. Dobrushin, R.A. Minlos, M.A. Shubin, A.M. Vershik, eds.), American Mathematical Society Translations, Ser. 2, Vol. 175, Amer. Math. Soc., Providence, 1996, pp. 137-175.

[12] M. Reed and B. Simon, *Methods of modern mathematical physics*, Vol. 1, 1980, ACADEMIC PRESS, INS.

[13] E.Thoma, *Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen symmetrischen Gruppe*, Math. Zeitschr. **85** (1964), no.1, 40-61.

[14] Takesaki M., *Theory of Operator Algebras, v.* I, Springer, 2005, 415 pp.

[15] Takesaki M., *Theory of Operator Algebras, v.* II, Springer, 2005, 518 pp.

A.V. Dudko, Department of Mathematics, University of Toronto,
e-mail: artem.dudko@utoronto.ca

N.I. Nessonov, B.Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine,
e-mail: nessonov@ilt.kharkov.ua